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Errata and supplement to the paper "Higher-order asymptotic cumulants of Studentized estimators in covariance structures"

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This note is to supplement Ogasawara (2008) with some errata. In this note

$$\mathrm{E}[\{m_{abcd}-\mathrm{E}(m_{abcd})\}\{m_{efgh}-\mathrm{E}(m_{efgh})\};\, n^{-1}]$$

$$\mathrm{E}[\{m_{abcd}-\mathrm{E}(m_{abcd})\}(s_{ef}-\sigma_{ef})(s_{gh}-\sigma_{gh});\, n^{-2}]$$

and $\mathrm{E}\{(m_{abcd}-\sigma_{abcd})(s_{ef}-\sigma_{ef})(s_{gh}-\sigma_{gh});\, n^{-2}\}$

will be given in Lemmas 3, 4 and 5, respectively. The sample moment m_{abcd} was defined in (2.6) while s_{ab} is the usual unbiased sample variance i.e., $\sum_{i=1}^{N}(X_{ia}-\bar{X}_a)(X_{ib}-\bar{X}_b)/(N-1)$. Since $X_{ia}-\bar{X}_a = X_{ia}-\mathrm{E}(X_{ia})-\{\bar{X}_a-\mathrm{E}(\bar{X}_a)\}$, X_{ia} and $\bar{X}_a$ are redefined in this appendix as the deviations from their expectations. Let $S_{a_1a_2...a_k}=\sum_{i=1}^{N}X_{ia_1}X_{ia_2}...X_{ia_k}\ (k=1,2,...)$, then

$$m_{abcd}=\frac{1}{N}S_{abcd}-\frac{1}{N^2}\sum^{4}S_aS_{bcd}+\frac{1}{N^3}\sum^{6}S_aS_bS_{cd}-\frac{3}{N^4}S_aS_bS_cS_d,$$
$$s_{ef}=\frac{1}{N-1}S_{ef}-\frac{1}{N(N-1)}S_eS_f. \qquad \text{(A.1)}$$

The following Lemmas 1 and 2 will be used in Lemmas 3 and 4.

Lemma 1.

$$\mathrm{E}(m_{abcd})=\frac{1}{N^3}(N^2-3N+3)(N-1)\sigma_{abcd}+\frac{1}{N^3}(2N-3)(N-1)\sum^{3}\sigma_{ab}\sigma_{cd}$$
$$=\left(1-\frac{4}{N}+\frac{6}{N^2}\right)\sigma_{abcd}+\left(\frac{2}{N}-\frac{5}{N^2}\right)\sum^{3}\sigma_{ab}\sigma_{cd}+O(N^{-3}). \qquad \text{(A.2)}$$

Proof. Noting that X_{ia}'s are deviations from their expectations, from (A.1) we have (A.2). Q. E. D.

Lemma 1 can also be obtained by generalizing the univariate exact result given by mathStatica (Rose and Smith, 2002):

$$\mathrm{E}(m_{aaaa}) = \frac{1}{N^3}(N^2 - 3N + 3)(N-1)\sigma_{aaaa} + \frac{3}{N^2}(2N-3)(N-1)\sigma_{aa}^2 . \quad \text{(A.3)}$$

Lemma 2.

$$\begin{aligned}\mathrm{E}(m_{abcd}s_{ef}) &= \left(\frac{1}{N} - \frac{4}{N^2}\right)\sigma_{abcdef} + \left(1 - \frac{5}{N} + \frac{10}{N^2}\right)\sigma_{abcd}\sigma_{ef} + \frac{2}{N^2}\sum^{6}\sigma_{abef}\sigma_{cd} \\ &+ \frac{1}{N^2}\sum^{8}\sigma_{bcde}\sigma_{af} + \left(-\frac{1}{N} + \frac{4}{N^2}\right)\sum^{4}\sigma_{aef}\sigma_{bcd} + \left(\frac{2}{N} - \frac{9}{N^2}\right)\sum^{3}\sigma_{ab}\sigma_{cd}\sigma_{ef} \\ &- \frac{1}{N^2}\sum^{6}(\sigma_{ae}\sigma_{bf} + \sigma_{af}\sigma_{be})\sigma_{cd} + O(N^{-3}).\end{aligned} \quad \text{(A.4)}$$

Proof. From (A.1), taking terms up to order $O(n^{-2})$,

$$\begin{aligned}\mathrm{E}(m_{abcd}s_{ef}) &= \frac{1}{N(N-1)}\{N\sigma_{abcdef} + (N^2 - N)\sigma_{abcd}\sigma_{ef}\} \\ &- \frac{1}{N^2(N-1)}\{4N\sigma_{abcdef} + (N^2-N)4\sigma_{abcd}\sigma_{ef} + (N^2-N)\sum^{4}\sigma_{aef}\sigma_{bcd}\} \\ &+ \frac{1}{N^3(N-1)}\{(N^2-N)(6\sigma_{abcd}\sigma_{ef} + 2\sum^{6}\sigma_{abef}\sigma_{cd} + 3\sum^{4}\sigma_{aef}\sigma_{bcd}) \\ &\qquad + N(N-1)(N-2)2\sigma_{ef}\sum^{3}\sigma_{ab}\sigma_{cd}\} \\ &- \frac{3}{N^4(N-1)}N(N-1)(N-2)\sigma_{ef}\sum^{3}\sigma_{ab}\sigma_{cd} \\ &- \frac{1}{N^2(N-1)}\{N\sigma_{abcdef} + (N^2-N)\sigma_{abcd}\sigma_{ef}\} \\ &+ \frac{1}{N^3(N-1)}(N^2-N)(4\sigma_{abcd}\sigma_{ef} + \sum^{8}\sigma_{bcde}\sigma_{af} + \sum^{4}\sigma_{aef}\sigma_{bcd}) \\ &- \frac{1}{N^4(N-1)}N(N-1)(N-2)\{2\sigma_{ef}\sum^{3}\sigma_{ab}\sigma_{cd} + \sum^{6}(\sigma_{ae}\sigma_{bf} + \sigma_{af}\sigma_{be})\sigma_{cd}\} \\ &+ O(N^{-3})\end{aligned}$$

$$= \frac{1}{N-1}\left(1-\frac{4}{N}-\frac{1}{N}\right)\sigma_{abcdef} + \left(1-\frac{4}{N}+\frac{6}{N^2}-\frac{1}{N}+\frac{4}{N^2}\right)\sigma_{abcd}\sigma_{ef}$$

$$+\frac{2}{N^2}\sum^{6}\sigma_{abef}\sigma_{cd} + \frac{1}{N^2}\sum^{8}\sigma_{bcde}\sigma_{af} + \left(-\frac{1}{N}+\frac{3}{N^2}+\frac{1}{N^2}\right)\sum^{4}\sigma_{aef}\sigma_{bcd}$$

$$+\left(\frac{2(N-2)}{N^2}-\frac{3}{N^2}-\frac{2}{N^2}\right)\sigma_{ef}\sum^{3}\sigma_{ab}\sigma_{cd} - \frac{1}{N^2}\sum^{6}(\sigma_{ae}\sigma_{bf}+\sigma_{af}\sigma_{be})\sigma_{cd} \tag{A.5}$$

$$+O(N^{-3}),$$

which gives (A.4). Q. E. D.

Then, we have

Lemma 3.

$$\begin{aligned}&\mathrm{E}[\{m_{abcd}-\mathrm{E}(m_{abcd})\}\{m_{efgh}-\mathrm{E}(m_{efgh})\}]\\&=\frac{1}{N}\{\sigma_{abcdefgh}-\sum^{4}(\sigma_{efgha}\sigma_{bcd}+\sigma_{abcde}\sigma_{fgh})-\sigma_{abcd}\sigma_{efgh}\\&\quad+\sum^{4^2=16}\sigma_{bcd}\sigma_{fgh}\sigma_{ae}\}+O(N^{-2}).\end{aligned} \tag{A.6}$$

Proof. From (A.1) and Lemma 1,

$$\begin{aligned}&-\mathrm{E}(m_{abcd})\mathrm{E}(m_{efgh})\\&=-\left\{\left(1-\frac{4}{N}\right)\sigma_{abcd}+\frac{2}{N}\sum^{3}\sigma_{ab}\sigma_{cd}\right\}\left\{\left(1-\frac{4}{N}\right)\sigma_{efgh}+\frac{2}{N}\sum^{3}\sigma_{ef}\sigma_{gh}\right\}+O(N^{-2})\\&=\left(-1+\frac{8}{N}\right)\sigma_{abcd}\sigma_{efgh}-\frac{2}{N}\sigma_{abcd}\sum^{3}\sigma_{ef}\sigma_{gh}-\frac{2}{N}\sigma_{efgh}\sum^{3}\sigma_{ab}\sigma_{cd}+O(N^{-2}),\end{aligned}$$

$$\begin{aligned}\mathrm{E}(m_{abcd}m_{efgh})&=\frac{1}{N}\sigma_{abcdefgh}-\frac{1}{N}\sum^{4}(\sigma_{efgha}\sigma_{bcd}+\sigma_{abcde}\sigma_{fgh})+\left(1-\frac{9}{N}\right)\sigma_{abcd}\sigma_{efgh}\\&\quad+\frac{1}{N}\sum^{4^2=16}\sigma_{bcd}\sigma_{fgh}\sigma_{ae}+\frac{2}{N}\sum^{3}(\sigma_{abcd}\sigma_{ef}\sigma_{gh}+\sigma_{efgh}\sigma_{ab}\sigma_{cd})+O(N^{-2}),\end{aligned} \tag{A.7}$$

which gives (A.6). Q. E. D.

Lemma 4.

$$\mathrm{E}[\{m_{abcd}-\mathrm{E}(m_{abcd})\}(s_{ef}-\sigma_{ef})(s_{gh}-\sigma_{gh})]$$

$$=\frac{1}{N^2}\Bigg[\ \sigma_{abcdefgh}-(\sigma_{abcdef}\sigma_{gh}+\sigma_{abcdgh}\sigma_{ef})$$

$$-\sum^{4}(\sigma_{bcdef}\sigma_{agh}+\sigma_{bcdgh}\sigma_{aef}+\sigma_{aefgh}\sigma_{bcd})$$

$$-\sum^{4}\sigma_{abcde}\sigma_{fgh}-\sigma_{abcd}\sigma_{efgh}+2\sigma_{abcd}\sigma_{ef}\sigma_{gh}-\sum^{4}(\sigma_{aef}\sigma_{gh}+\sigma_{agh}\sigma_{ef})\sigma_{bcd}$$

$$+\sum^{4}(\sigma_{ag}\sigma_{efh}+\sigma_{ah}\sigma_{efg}+\sigma_{ae}\sigma_{ghf}+\sigma_{af}\sigma_{ghe})\sigma_{bcd}$$

$$+\sum^{{}_4C_2=6}\{(\sigma_{aef}\sigma_{bgh}+\sigma_{agh}\sigma_{bef})\sigma_{cd}+(\sigma_{acd}\sigma_{bgh}+\sigma_{agh}\sigma_{bcd})\sigma_{ef}$$

$$+(\sigma_{acd}\sigma_{bef}+\sigma_{aef}\sigma_{bcd})\sigma_{gh}\}\ \Bigg]+O(N^{-3}).$$

(A.8)

Proof. Decompose the expectation in (A.8) as

$$\mathrm{E}(m_{abcd}s_{ef}s_{gh})-\mathrm{E}(m_{abcd})\mathrm{E}\{(s_{ef}-\sigma_{ef})(s_{gh}-\sigma_{gh})\}-\mathrm{E}(m_{abcd}s_{ef})\sigma_{gh}$$
$$-\mathrm{E}(m_{abcd}s_{gh})\sigma_{ef}+\mathrm{E}(m_{abcd})\sigma_{ef}\sigma_{gh}. \quad \text{(A.9)}$$

We derive the above expectations one by one in the following Sections a to d.

a. $\mathrm{E}(m_{abcd}s_{ef}s_{gh})$

Since

$$m_{abcd}s_{ef}s_{gh}=\frac{m_{abcd}}{(N-1)^2}S_{ef}S_{gh}-\frac{m_{abcd}}{(N-1)(N^2-N)}S_{ef}S_gS_h$$
$$-\frac{m_{abcd}}{(N-1)(N^2-N)}S_{gh}S_eS_f+\frac{m_{abcd}}{(N^2-N)^2}S_eS_fS_gS_h, \quad \text{(A.10)}$$

we take the expectations of the four terms on the right-hand side of (A.10) in the following Subsections a.1 to a.4.

a.1 $\mathrm{E}\{m_{abcd}S_{ef}S_{gh}/(N-1)^2\}$

$$\mathrm{E}\left\{\frac{m_{abcd}}{(N-1)^2}S_{ef}S_{gh}\right\}$$

$$=\frac{1}{N(N-1)^2}\{N\sigma_{abcdefgh}+(N^2-N)(\sigma_{abcdef}\sigma_{gh}+\sigma_{abcdgh}\sigma_{ef}$$

$$+\sigma_{abcd}\sigma_{efgh})+N(N-1)(N-2)\sigma_{abcd}\sigma_{ef}\sigma_{gh}\}-\frac{1}{N^2(N-1)^2}\Big[\ (N^2-N)$$

$$\times 4(\sigma_{abcdef}\sigma_{gh}+\sigma_{abcdgh}\sigma_{ef})+(N^2-N)\sum^{4}(\sigma_{bcdef}\sigma_{agh}+\sigma_{bcdgh}\sigma_{aef}+\sigma_{aefgh}\sigma_{bcd})$$

$$+(N^2-N)4\sigma_{abcd}\sigma_{efgh}$$

$$+N(N-1)(N-2)\{4\sigma_{abcd}\sigma_{ef}\sigma_{gh}+\sum^{4}(\sigma_{aef}\sigma_{bcd}\sigma_{gh}+\sigma_{agh}\sigma_{bcd}\sigma_{ef})\}\Big]$$

$$+\frac{1}{N^3(N-1)^2}\Big[\ N(N-1)(N-2)\{\ 6\sigma_{abcd}\sigma_{ef}\sigma_{gh}+\sigma_{efgh}2\sum^{3}\sigma_{ab}\sigma_{cd}$$

$$+\sum^{{}_4C_2=6}(\sigma_{aef}\sigma_{bgh}\sigma_{cd}+\sigma_{agh}\sigma_{bef}\sigma_{cd}$$

$$+\sigma_{acd}\sigma_{bef}\sigma_{gh}+\sigma_{acd}\sigma_{bgh}\sigma_{ef}+\sigma_{bcd}\sigma_{aef}\sigma_{gh}+\sigma_{bcd}\sigma_{agh}\sigma_{ef})$$

$$+2\sum^{6}(\sigma_{cdef}\sigma_{ab}\sigma_{gh}+\sigma_{cdgh}\sigma_{ab}\sigma_{ef})\ \}$$

$$+N(N-1)(N-2)(N-3)\sigma_{ef}\sigma_{gh}2\sum^{3}\sigma_{ab}\sigma_{cd}\Big]$$

$$-\frac{1}{N^4(N-1)^2}N(N-1)(N-2)(N-3)3\sigma_{ef}\sigma_{gh}\sum^{3}\sigma_{ab}\sigma_{cd}+O(N^{-3})$$

$$=\frac{1}{N^2}\sigma_{abcdefgh}+\left(\frac{1}{N-1}-\frac{4}{N^2}\right)(\sigma_{abcdef}\sigma_{gh}+\sigma_{abcdgh}\sigma_{ef})$$

$$-\frac{1}{N^2}\sum^{4}(\sigma_{bcdef}\sigma_{agh}+\sigma_{bcdgh}\sigma_{aef}$$

$$+\sigma_{aefgh}\sigma_{bcd})+\left(\frac{1}{N-1}-\frac{4}{N^2}\right)\sigma_{abcd}\sigma_{efgh}$$

$$+\left(\frac{N-2}{N-1}-\frac{N-2}{N(N-1)}4+\frac{N-2}{N^2(N-1)}6\right)\sigma_{abcd}\sigma_{ef}\sigma_{gh}$$

$$+\frac{N-2}{N^2(N-1)}\sigma_{efgh}2\sum^{3}\sigma_{ab}\sigma_{cd}+\frac{N-2}{N^2(N-1)}2\sum^{6}(\sigma_{cdef}\sigma_{ab}\sigma_{gh}+\sigma_{cdgh}\sigma_{ab}\sigma_{ef})$$

$$-\frac{N-2}{N(N-1)}\sum^{4}(\sigma_{aef}\sigma_{bcd}\sigma_{gh}+\sigma_{agh}\sigma_{bcd}\sigma_{ef})$$

$$+\frac{N-2}{N^2(N-1)}\sum^{6}(\sigma_{aef}\sigma_{bgh}\sigma_{cd}+\sigma_{agh}\sigma_{bef}\sigma_{cd}$$

$$+\sigma_{acd}\sigma_{bef}\sigma_{gh}+\sigma_{acd}\sigma_{bgh}\sigma_{ef}+\sigma_{bcd}\sigma_{aef}\sigma_{gh}+\sigma_{bcd}\sigma_{agh}\sigma_{ef})$$

$$+\left(\frac{(N-2)(N-3)}{N^2(N-1)}2-\frac{(N-2)(N-3)}{N^3(N-1)}3\right)\sigma_{ef}\sigma_{gh}\sum^{3}\sigma_{ab}\sigma_{cd}+O(N^{-3})$$

$$=\frac{1}{N^2}\sigma_{abcdefgh}+\left(\frac{1}{N}-\frac{3}{N^2}\right)(\sigma_{abcdef}\sigma_{gh}+\sigma_{abcdgh}\sigma_{ef})$$

$$-\frac{1}{N^2}\sum^{4}(\sigma_{bcdef}\sigma_{agh}+\sigma_{bcdgh}\sigma_{aef}+\sigma_{aefgh}\sigma_{bcd})$$

$$+\left(\frac{1}{N}-\frac{3}{N^2}\right)\sigma_{abcd}\sigma_{efgh}+\left(1-\frac{5}{N}+\frac{9}{N^2}\right)\sigma_{abcd}\sigma_{ef}\sigma_{gh}$$

$$+\frac{2}{N^2}\sigma_{efgh}\sum^{3}\sigma_{ab}\sigma_{cd}+\frac{2}{N^2}\sum^{6}(\sigma_{cdef}\sigma_{ab}\sigma_{gh}+\sigma_{cdgh}\sigma_{ab}\sigma_{ef})$$

$$+\left(-\frac{1}{N}+\frac{1}{N^2}\right)\sum^{4}(\sigma_{aef}\sigma_{bcd}\sigma_{gh}+\sigma_{agh}\sigma_{bcd}\sigma_{ef})$$

$$+\frac{1}{N^2}\sum^{6}(\sigma_{aef}\sigma_{bgh}\sigma_{cd}+\sigma_{agh}\sigma_{bef}\sigma_{cd}$$
$$+\sigma_{acd}\sigma_{bef}\sigma_{gh}+\sigma_{acd}\sigma_{bgh}\sigma_{ef}+\sigma_{bcd}\sigma_{aef}\sigma_{gh}+\sigma_{bcd}\sigma_{agh}\sigma_{ef}) \tag{A.11}$$
$$+\left(\frac{2}{N}-\frac{11}{N^2}\right)\sigma_{ef}\sigma_{gh}\sum^{3}\sigma_{ab}\sigma_{cd}+O(N^{-3}).$$

a.2 $-\mathrm{E}[m_{abcd}S_{ef}S_gS_h/\{(N-1)(N^2-N)\}]$

$$-\mathrm{E}\left\{\frac{m_{abcd}}{(N-1)(N^2-N)}S_{ef}S_gS_h\right\}$$
$$=-\frac{1}{N^2(N-1)^2}\{N(N-1)(\sigma_{abcdef}\sigma_{gh}+\sigma_{abcdgh}\sigma_{ef})$$
$$+N(N-1)(\sigma_{abcdg}\sigma_{efh}+\sigma_{abcdh}\sigma_{efg})+N(N-1)\sigma_{abcd}\sigma_{efgh}$$
$$+N(N-1)(N-2)\sigma_{abcd}\sigma_{ef}\sigma_{gh}\}$$
$$+\frac{1}{N^3(N-1)^2}N(N-1)(N-2)\Big\{4\sigma_{abcd}\sigma_{ef}\sigma_{gh}$$
$$+\sum^{4}(\sigma_{bcdg}\sigma_{ah}\sigma_{ef}+\sigma_{bcdh}\sigma_{ag}\sigma_{ef})$$
$$+\sum^{4}(\sigma_{ag}\sigma_{efh}+\sigma_{ah}\sigma_{efg}+\sigma_{aef}\sigma_{gh}+\sigma_{ef}\sigma_{agh})\sigma_{bcd}\Big\}$$
$$-\frac{1}{N^4(N-1)^2}N(N-1)(N-2)(N-3)$$
$$\times\sum^{{}_4C_2=6}(\sigma_{ag}\sigma_{bh}+\sigma_{ah}\sigma_{bg}+\sigma_{ab}\sigma_{gh})\sigma_{cd}\sigma_{ef}+O(N^{-3})$$

$$= -\frac{1}{N^2}(\sigma_{abcdef}\sigma_{gh} + \sigma_{abcdgh}\sigma_{ef}) - \frac{1}{N^2}(\sigma_{abcdg}\sigma_{efh} + \sigma_{abcdh}\sigma_{efg})$$

$$-\frac{1}{N^2}\sigma_{abcd}\sigma_{efgh}$$

$$+\left(-\frac{1}{N} + \frac{5}{N^2}\right)\sigma_{abcd}\sigma_{ef}\sigma_{gh} + \frac{1}{N^2}\sum^{4}(\sigma_{bcdg}\sigma_{ah}\sigma_{ef} + \sigma_{bcdh}\sigma_{ag}\sigma_{ef})$$

$$+\frac{1}{N^2}\sum^{4}(\sigma_{ag}\sigma_{efh} + \sigma_{ah}\sigma_{efg} + \sigma_{aef}\sigma_{gh} + \sigma_{ef}\sigma_{agh})\sigma_{bcd}$$

$$-\frac{1}{N^2}\sum^{6}(\sigma_{ag}\sigma_{bh} + \sigma_{ah}\sigma_{bg} + \sigma_{ab}\sigma_{gh})\sigma_{cd}\sigma_{ef} + O(N^{-3}).$$

(A.12)

a.3 $-\mathrm{E}[m_{abcd}S_{gh}S_eS_f/\{(N-1)(N^2-N)\}]$

The result can be obtained from Subsection a.2 by exchanging subscripts *e* and *f* with *g* and *h*, respectively.

a.4 $\mathrm{E}\{m_{abcd}S_eS_fS_gS_h/(N^2-N)^2\}$

$$\mathrm{E}\left\{\frac{m_{abcd}}{(N^2-N)^2}S_eS_fS_gS_h\right\}$$

$$= \frac{1}{N^3(N-1)^2}N(N-1)(N-2)(\sigma_{ef}\sigma_{gh} + \sigma_{eg}\sigma_{fh} + \sigma_{eh}\sigma_{fg})\sigma_{abcd} + O(N^{-3})$$

$$= \frac{1}{N^2}(\sigma_{ef}\sigma_{gh} + \sigma_{eg}\sigma_{fh} + \sigma_{eh}\sigma_{fg})\sigma_{abcd} + O(N^{-3}).$$

(A.13)

From Subsections a.1 through a.4 with (A.10), we have

$$\mathrm{E}(m_{abcd}s_{ef}s_{gh}) = \frac{1}{N^2}\sigma_{abcdefgh} + \left(\frac{1}{N} - \frac{5}{N^2}\right)(\sigma_{abcdef}\sigma_{gh} + \sigma_{abcdgh}\sigma_{ef})$$

$$-\frac{1}{N^2}\left\{ \sum^{4}(\sigma_{bcdef}\sigma_{agh} + \sigma_{bcdgh}\sigma_{aef} + \sigma_{aefgh}\sigma_{bcd}) + \sum^{4}\sigma_{abcde}\sigma_{fgh} \right\}$$

$$+\left(\frac{1}{N} - \frac{5}{N^2}\right)\sigma_{abcd}\sigma_{efgh} + \left(1 - \frac{7}{N} + \frac{20}{N^2}\right)\sigma_{abcd}\sigma_{ef}\sigma_{gh} + \frac{2}{N^2}\sigma_{efgh}\sum^{3}\sigma_{ab}\sigma_{cd}$$

$$+\frac{1}{N^2}\Bigg[\sigma_{abcd}(\sigma_{eg}\sigma_{fh} + \sigma_{eh}\sigma_{fg}) + 2\sum^{6}(\sigma_{cdef}\sigma_{gh} + \sigma_{cdgh}\sigma_{ef})\sigma_{ab}$$

$$+\sum^{4}\{(\sigma_{bcdg}\sigma_{ah} + \sigma_{bcdh}\sigma_{ag})\sigma_{ef}$$

$$+(\sigma_{bcde}\sigma_{af} + \sigma_{bcdf}\sigma_{ae})\sigma_{gh}\}\Bigg] + \left(-\frac{1}{N} + \frac{3}{N^2}\right)\sum^{4}(\sigma_{aef}\sigma_{gh} + \sigma_{agh}\sigma_{ef})\sigma_{bcd}$$

$$+\frac{1}{N^2}\sum^{4}(\sigma_{ag}\sigma_{efh} + \sigma_{ah}\sigma_{efg} + \sigma_{ae}\sigma_{ghf} + \sigma_{af}\sigma_{ghe})\sigma_{bcd}$$

$$+\frac{1}{N^2}\sum^{6}\{(\sigma_{aef}\sigma_{bgh} + \sigma_{agh}\sigma_{bef})\sigma_{cd} + (\sigma_{acd}\sigma_{bgh} + \sigma_{agh}\sigma_{bcd})\sigma_{ef}$$

$$+(\sigma_{acd}\sigma_{bef} + \sigma_{aef}\sigma_{bcd})\sigma_{gh}\} + \left(\frac{2}{N} - \frac{15}{N^2}\right)\sigma_{ef}\sigma_{gh}\sum^{3}\sigma_{ab}\sigma_{cd}$$

$$-\frac{1}{N^2}\sum^{6}\{(\sigma_{ag}\sigma_{bh} + \sigma_{ah}\sigma_{bg})\sigma_{ef} + (\sigma_{ae}\sigma_{bf} + \sigma_{af}\sigma_{be})\sigma_{gh}\}\sigma_{cd} + O(N^{-3}). \tag{A.14}$$

b. $-\mathrm{E}(m_{abcd})\mathrm{E}\{(s_{ef} - \sigma_{ef})(s_{gh} - \sigma_{gh})\}$

$$-\mathrm{E}(m_{abcd})\mathrm{E}\{(s_{ef} - \sigma_{ef})(s_{gh} - \sigma_{gh})\} = \left(-\frac{1}{N} + \frac{4}{N^2}\right)\sigma_{abcd}\sigma_{efgh}$$

$$+\left(\frac{1}{N} - \frac{4}{N^2}\right)\sigma_{abcd}\sigma_{ef}\sigma_{gh} - \frac{1}{N^2}\sigma_{abcd}(\sigma_{eg}\sigma_{fh} + \sigma_{eh}\sigma_{fg}) - \frac{2}{N^2}\sigma_{efgh}\sum^{3}\sigma_{ab}\sigma_{cd}$$

$$+\frac{2}{N^2}\sigma_{ef}\sigma_{gh}\sum^{3}\sigma_{ab}\sigma_{cd} + O(N^{-3}), \tag{A.15}$$

where

$$\mathrm{E}\{(s_{ef}-\sigma_{ef})(s_{gh}-\sigma_{gh})\}=\frac{1}{N}(\sigma_{efgh}-\sigma_{ef}\sigma_{gh})+\frac{1}{N(N-1)}(\sigma_{eg}\sigma_{fh}+\sigma_{eh}\sigma_{fg})$$
$$=\frac{1}{N}(\sigma_{efgh}-\sigma_{ef}\sigma_{gh})+\frac{1}{N^2}(\sigma_{eg}\sigma_{fh}+\sigma_{eh}\sigma_{fg})+O(N^{-3}) \tag{A.16}$$

(see e.g., Kaplan, 1952) is used.

c. $-\mathrm{E}(m_{abcd}s_{ef})\sigma_{gh}-\mathrm{E}(m_{abcd}s_{gh})\sigma_{ef}$

$$-\mathrm{E}(m_{abcd}s_{ef})\sigma_{gh}-\mathrm{E}(m_{abcd}s_{gh})\sigma_{ef}$$
$$=\left(-\frac{1}{N}+\frac{4}{N^2}\right)(\sigma_{abcdef}\sigma_{gh}+\sigma_{abcdgh}\sigma_{ef})$$
$$+\left(-2+\frac{10}{N}-\frac{20}{N^2}\right)\sigma_{abcd}\sigma_{ef}\sigma_{gh}-\frac{2}{N^2}\sum^{6}(\sigma_{abef}\sigma_{gh}+\sigma_{abgh}\sigma_{ef})\sigma_{cd}$$
$$-\frac{1}{N^2}\sum^{8}(\sigma_{bcde}\sigma_{af}\sigma_{gh}+\sigma_{bcdg}\sigma_{ah}\sigma_{ef})$$
$$+\left(\frac{1}{N}-\frac{4}{N^2}\right)\sum^{4}(\sigma_{aef}\sigma_{bcd}\sigma_{gh}+\sigma_{agh}\sigma_{bcd}\sigma_{ef}) \tag{A.17}$$
$$+\left(-\frac{4}{N}+\frac{18}{N^2}\right)\sigma_{ef}\sigma_{gh}\sum^{3}\sigma_{ab}\sigma_{cd}+\frac{1}{N^2}\sum^{6}\{(\sigma_{ae}\sigma_{bf}+\sigma_{af}\sigma_{be})\sigma_{gh}$$
$$+(\sigma_{ag}\sigma_{bh}+\sigma_{ah}\sigma_{bg})\sigma_{ef}\}\sigma_{cd}.$$

d. $\mathrm{E}(m_{abcd})\sigma_{ef}\sigma_{gh}$

$$\mathrm{E}(m_{abcd})\sigma_{ef}\sigma_{gh}=\left(1-\frac{4}{N}+\frac{6}{N^2}\right)\sigma_{abcd}\sigma_{ef}\sigma_{gh}$$
$$+\left(\frac{2}{N}-\frac{5}{N^2}\right)\sigma_{ef}\sigma_{gh}\sum^{3}\sigma_{ab}\sigma_{cd}+O(N^{-3}). \tag{A.18}$$

Summing (A.14), (A.15), (A.17) and (A.18), (A.8) follows. Q. E. D.

From Lemmas 1 and 4, we have

Lemma 5.

$$\mathrm{E}\{(m_{abcd}-\sigma_{abcd})(s_{ef}-\sigma_{ef})(s_{gh}-\sigma_{gh})\}$$

$$=\frac{1}{N^2}\Bigg[\ \sigma_{abcdefgh}-(\sigma_{abcdef}\sigma_{gh}+\sigma_{abcdgh}\sigma_{ef})$$

$$-\sum^{4}(\sigma_{bcdef}\sigma_{agh}+\sigma_{bcdgh}\sigma_{aef}+\sigma_{aefgh}\sigma_{bcd})$$

$$-\sum^{4}\sigma_{abcde}\sigma_{fgh}-5\sigma_{abcd}\sigma_{efgh}+6\sigma_{abcd}\sigma_{ef}\sigma_{gh}-\sum^{4}(\sigma_{aef}\sigma_{gh}+\sigma_{agh}\sigma_{ef})\sigma_{bcd}$$

$$+\sum^{4}(\sigma_{ag}\sigma_{efh}+\sigma_{ah}\sigma_{efg}+\sigma_{ae}\sigma_{ghf}+\sigma_{af}\sigma_{ghe})\sigma_{bcd}$$

$$+\sum^{{}_4C_2=6}\{(\sigma_{aef}\sigma_{bgh}+\sigma_{agh}\sigma_{bef})\sigma_{cd}+(\sigma_{acd}\sigma_{bgh}+\sigma_{agh}\sigma_{bcd})\sigma_{ef}$$

$$+(\sigma_{acd}\sigma_{bef}+\sigma_{aef}\sigma_{bcd})\sigma_{gh}\}+2\sum^{3}\sigma_{ab}\sigma_{cd}(\sigma_{efgh}-\sigma_{ef}\sigma_{gh})\Bigg]+O(N^{-3}). \tag{A.19}$$

For the asymptotic cumulants of the Studentized parameter estimators, Lemma 5 should have been used in place of Lemma 4 when m_{abcd} is evaluated at σ_{abcd} while the other asymptotic results e.g., Lemma 3 hold even when m_{abcd} is evaluated at $\mathrm{E}(m_{abcd})$. The vector $\mathbf{u}_{(4)}$ should have been defined as $\mathbf{u}_{(4)}=n^{1/2}\{(\mathbf{s}-\boldsymbol{\sigma})',(\mathbf{m}_{(4)}-\boldsymbol{\sigma}_{(4)})'\}'$, where $\boldsymbol{\sigma}_{(4)}$ is the population counterpart of $\mathbf{m}_{(4)}$.

The numerical results in Tables 1 and 2 have been corrected and are presented as Tables 1A and 2A, respectively.

References

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Table 1A. Simulated and theoretical cumulants of Studentized estimators in bivariate data

	Regression coefficient					Residual variance				
	Nml	U	T9	C10	C3	Nml	U	T9	C10	C3
(*N*)	$(1+n^{-1}\Delta\alpha_2')^{1/2}$: higher-order asymptotic standard error									
(51)	1.110	1.084	1.121	1.123	1.155	1.246	1.092	1.385	1.446	1.807
Th.	1.105	1.074	1.136	1.145	1.303	1.227	1.143	1.680	1.564	1.990
(201)	1.028	1.020	1.033	1.035	1.054	1.059	1.019	1.116	1.135	1.237
Th.	1.027	1.019	1.036	1.038	1.084	1.061	1.037	1.207	1.167	1.319
(801)	1.005	1.005	1.009	1.010	1.017	1.015	1.005	1.035	1.040	1.073
Th.	1.007	1.005	1.009	1.010	1.022	1.016	1.009	1.055	1.044	1.089
(*N*)	$\Delta\alpha_2'$: higher-order added variance									
(51)	11.7	8.7	12.9	13.1	16.7	27.6	9.6	46.0	54.6	113.3
(201)	11.2	7.9	13.3	14.2	22.1	24.2	7.7	48.9	57.7	105.9
(801)	8.6	7.8	14.0	15.3	28.1	23.5	8.5	57.7	65.0	120.5
Th.	11.1	7.7	14.5	15.6	34.9	25.3	15.3	91.2	72.4	147.9
(*N*)	α_1': bias									
(51)	-.01	-.01	-.00	-.21	-.67	-2.47	-1.55	-3.46	-3.64	-5.27
(201)	.01	-.00	.00	-.32	-1.00	-2.20	-1.47	-3.33	-3.38	-4.65
(801)	.04	-.03	.02	-.36	-1.22	-2.15	-1.47	-3.45	-3.38	-4.57
Th.	0	0	0	-.40	-1.33	-2.12	-1.44	-3.91	-3.35	-4.67
(*N*)	α_3': skewness									
(51)	-.03	-.02	.02	-.39	-.89	-13.01	-2.58	-22.63	-27.22	-78.60
(201)	-.04	.02	-.03	-1.08	-3.36	-7.10	-1.50	-12.80	-14.74	-27.49
(801)	.05	-.06	.03	-1.50	-4.51	-6.00	-1.42	-11.62	-12.16	-19.28
Th.	0	0	0	-1.60	-5.33	-5.66	-1.28	-13.42	-11.46	-17.42
(*N*)	α_4': kurtosis									
(51)	27	29	26	26	24	245	34	502	637	3449
(201)	19	19	16	23	24	105	16	229	315	907
(801)	17	14	19	16	40	77	15	216	275	599
Th.	30	29	26	34	80	110	72	145	313	657

Note. *N*=*n*+1=Sample size; (*N*)=Simulated values with *N* in the simulation; Th.=Theoretical values; Nml, U, T9, C10 and C3=Normal, uniform, *t*- (*df*=9) and chi-square (*df*=10, 3) distributions, respectively.

Table 2A. $10^5 \times$ root mean square errors of the asymptotic distribution functions of the Studentized estimators in bivariate data

N	Method	Nml	U	T9	C10	C3
		Data				
	Regression coefficient					
	N*	1152	797	1314	1415	2356
51	E1	1152	797	1314	1390	2307
	E2	213	253	229	486	2290
	Hall	1152	797	1314	1391	2425
	N*	296	195	372	523	1256
201	E1	296	195	372	392	723
	E2	67	64	35	93	459
	Hall	296	195	372	393	753
	N*	60	60	101	225	690
801	E1	60	60	101	113	208
	E2	31	36	21	31	75
	Hall	60	60	101	113	216
	Residual variance					
	N*	5343	3756	7297	7603	10074
51	E1	2196	1083	3948	3841	5983
	E2	841	765	8422	3149	8074
	Hall	2001	906	4602	3958	6390
	N*	2435	1814	3657	3682	4929
201	E1	563	244	1351	1229	2036
	E2	158	198	1954	511	1497
	Hall	524	199	1542	1237	2216
	N*	1199	905	1898	1848	2487
801	E1	148	70	435	368	653
	E2	42	47	468	76	235
	Hall	141	59	473	366	682

Note. N*=Norma approximation, E1=The single-term Edgeworth expansion, E2=The two-term Edgeworth expansion, Hall=Hall's method by variable transformation; Nml, U, T9, C10 and C3=Normal, uniform, *t*- (*df*=9) and chi-square (*df*=10, 3) distributions, respectively.