

**A supplementary note on the paper
“Higher-order asymptotic standard error and asymptotic
expansion in principal component analysis”**

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This note is to supplement Ogasawara (2006) as an appendix, which was not included in the article due to space limitation. Section 1 for the formulas of the partial derivatives in implicit functions was first given by Ogasawara (2005a). Section 2 gives the partial derivatives of the discrepancy function in principal component analysis with respect to parameters and sample covariances required in using the results of Section 1.

1. The formulas of the partial derivatives of $\hat{\theta}$ with respect to s (Ogasawara, 2005a)

$$\text{Let } \hat{\mathbf{H}}_A = \begin{bmatrix} \frac{\partial^2 \hat{F}}{\partial \hat{\boldsymbol{\theta}} \partial \hat{\boldsymbol{\theta}}'} & \frac{\partial \hat{\mathbf{h}}'}{\partial \hat{\boldsymbol{\theta}}} \\ \frac{\partial \hat{\mathbf{h}}}{\partial \hat{\boldsymbol{\theta}}'} & \mathbf{O} \end{bmatrix}, \text{ then}$$

$$\begin{bmatrix} \frac{\partial \hat{\boldsymbol{\theta}}}{\partial s_{ab}} \\ \mathbf{0} \end{bmatrix} = -\hat{\mathbf{H}}_A^{-1} \begin{bmatrix} \frac{\partial^2 \hat{F}}{\partial \hat{\boldsymbol{\theta}} \partial s_{ab}} \\ \mathbf{0} \end{bmatrix}, \quad (\text{A1})$$

$$\begin{aligned}
\begin{bmatrix} \frac{\partial^2 \hat{\boldsymbol{\theta}}}{\partial s_{ab} \partial s_{cd}} \\ \mathbf{0} \end{bmatrix} &= -\hat{\mathbf{H}}_{\mathbf{A}}^{-1} \left\{ \sum_{i,j=1}^q \left[\frac{\frac{\partial^3 \hat{F}}{\partial \hat{\boldsymbol{\theta}} \partial \hat{\theta}_i \partial \hat{\theta}_j}}{\frac{\partial^2 \hat{\mathbf{h}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j}} \right] \frac{\partial \hat{\theta}_i}{\partial s_{ab}} \frac{\partial \hat{\theta}_j}{\partial s_{cd}} + \sum_{i=1}^q \left[\frac{\frac{\partial^3 \hat{F}}{\partial \hat{\boldsymbol{\theta}} \partial \hat{\theta}_i \partial s_{cd}}}{\mathbf{0}} \right] \frac{\partial \hat{\theta}_i}{\partial s_{ab}} \right. \\
&\quad \left. + \sum_{i=1}^q \left[\frac{\frac{\partial^3 \hat{F}}{\partial \hat{\boldsymbol{\theta}} \partial \hat{\theta}_i \partial s_{ab}}}{\mathbf{0}} \right] \frac{\partial \hat{\theta}_i}{\partial s_{cd}} + \left[\frac{\frac{\partial^3 \hat{F}}{\partial \hat{\boldsymbol{\theta}} \partial s_{cd} \partial s_{ab}}}{\mathbf{0}} \right] \right\}
\end{aligned} \tag{A2}$$

and

$$\begin{aligned}
\begin{bmatrix} \frac{\partial^3 \hat{\boldsymbol{\theta}}}{\partial s_{ab} \partial s_{cd} \partial s_{ef}} \\ \mathbf{0} \end{bmatrix} &= -\hat{\mathbf{H}}_{\mathbf{A}}^{-1} \left\{ \sum_{i,j,k=1}^q \left[\frac{\frac{\partial^4 \hat{F}}{\partial \hat{\boldsymbol{\theta}} \partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k}}{\frac{\partial^3 \hat{\mathbf{h}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k}} \right] \frac{\partial \hat{\theta}_i}{\partial s_{ab}} \frac{\partial \hat{\theta}_j}{\partial s_{cd}} \frac{\partial \hat{\theta}_k}{\partial s_{ef}} \right. \\
&\quad + \sum_{(U,V,W)}^3 \sum_{i,j=1}^q \left[\frac{\frac{\frac{\partial^3 \hat{F}}{\partial \hat{\boldsymbol{\theta}} \partial \hat{\theta}_i \partial \hat{\theta}_j}}{\frac{\partial^2 \hat{\mathbf{h}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j}}}{\frac{\partial^2 \hat{\mathbf{h}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j}} \right] \frac{\partial \hat{\theta}_i}{\partial s_U} \frac{\partial^2 \hat{\theta}_j}{\partial s_V \partial s_W} \\
&\quad + \sum_{(U,V,W)}^3 \sum_{i,j=1}^q \left[\frac{\frac{\frac{\partial^4 \hat{F}}{\partial \hat{\boldsymbol{\theta}} \partial \hat{\theta}_i \partial \hat{\theta}_j \partial s_U}}{\mathbf{0}}}{\mathbf{0}} \right] \frac{\partial \hat{\theta}_i}{\partial s_V} \frac{\partial \hat{\theta}_j}{\partial s_W} \\
&\quad + \sum_{(U,V,W)}^3 \sum_{i=1}^q \left[\frac{\frac{\frac{\partial^3 \hat{F}}{\partial \hat{\boldsymbol{\theta}} \partial \hat{\theta}_i \partial s_U}}{\mathbf{0}}}{\mathbf{0}} \right] \frac{\partial^2 \hat{\theta}_i}{\partial s_V \partial s_W} + \sum_{(U,V,W)}^3 \sum_{i=1}^q \left[\frac{\frac{\frac{\partial^4 \hat{F}}{\partial \hat{\boldsymbol{\theta}} \partial \hat{\theta}_i \partial s_U \partial s_V}}{\mathbf{0}}}{\mathbf{0}} \right] \frac{\partial \hat{\theta}_i}{\partial s_W} \\
&\quad \left. + \left[\frac{\frac{\partial^4 \hat{F}}{\partial \hat{\boldsymbol{\theta}} \partial s_{ab} \partial s_{cd} \partial s_{ed}}}{\mathbf{0}} \right] \right\}, \quad (p \geq a \geq b \geq 1; p \geq c \geq d \geq 1; p \geq e \geq f \geq 1),
\end{aligned}$$

(A3)

where $\sum_{(U,V,W)}^3$ denotes the summation over the range $(U, V, W) \in \{(ab, cd, ef), (cd, ef, ab), (ef, ab, cd)\}$.

2. The partial derivatives of the discrepancy functions with respect to the parameters and sample variances and covariances

2.1 $F = (1/2)\text{tr}\{(\mathbf{\Sigma}(\boldsymbol{\theta}) - \mathbf{S})^2\}$

Let δ_{ab} be the Kronecker delta and $\hat{\sigma}_{ab} = (\hat{\mathbf{\Sigma}})_{ab} = \{\mathbf{\Sigma}(\hat{\boldsymbol{\theta}})\}_{ab}$, then

$$\frac{\partial^2 \hat{F}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} = \text{tr} \left\{ \frac{\partial \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_i} \frac{\partial \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_j} + (\hat{\mathbf{\Sigma}} - \mathbf{S}) \frac{\partial^2 \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \right\}, \quad (\text{A4})$$

$$\frac{\partial^2 \hat{F}}{\partial \hat{\theta}_i \partial s_{ab}} = -(2 - \delta_{ab}) \frac{\partial \hat{\sigma}_{ab}}{\partial \hat{\theta}_i}, \quad (\text{A5})$$

$$\begin{aligned} \frac{\partial^3 \hat{F}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k} = \text{tr} \left\{ \frac{\partial \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_i} \frac{\partial^2 \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_j \partial \hat{\theta}_k} + \frac{\partial \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_j} \frac{\partial^2 \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_i \partial \hat{\theta}_k} \right. \\ \left. + \frac{\partial \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_k} \frac{\partial^2 \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} + (\hat{\mathbf{\Sigma}} - \mathbf{S}) \frac{\partial^3 \hat{\mathbf{\Sigma}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k} \right\}, \quad (\text{A6}) \end{aligned}$$

$$\frac{\partial^3 \hat{F}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial s_{ab}} = -(2 - \delta_{ab}) \frac{\partial^2 \hat{\sigma}_{ab}}{\partial \hat{\theta}_i \partial \hat{\theta}_j}, \quad (\text{A7})$$

$$\begin{aligned}
\frac{\partial^4 \hat{F}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k \partial \hat{\theta}_l} = & \text{tr} \left\{ \frac{\partial \hat{\Sigma}}{\partial \hat{\theta}_i} \frac{\partial^3 \hat{\Sigma}}{\partial \hat{\theta}_j \partial \hat{\theta}_k \partial \hat{\theta}_l} + \frac{\partial \hat{\Sigma}}{\partial \hat{\theta}_j} \frac{\partial^3 \hat{\Sigma}}{\partial \hat{\theta}_i \partial \hat{\theta}_k \partial \hat{\theta}_l} \right. \\
& + \frac{\partial \hat{\Sigma}}{\partial \hat{\theta}_k} \frac{\partial^3 \hat{\Sigma}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_l} + \frac{\partial \hat{\Sigma}}{\partial \hat{\theta}_l} \frac{\partial^3 \hat{\Sigma}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k} + \frac{\partial^2 \hat{\Sigma}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \frac{\partial^2 \hat{\Sigma}}{\partial \hat{\theta}_k \partial \hat{\theta}_l} \\
& + \frac{\partial^2 \hat{\Sigma}}{\partial \hat{\theta}_i \partial \hat{\theta}_k} \frac{\partial^2 \hat{\Sigma}}{\partial \hat{\theta}_j \partial \hat{\theta}_l} + \frac{\partial^2 \hat{\Sigma}}{\partial \hat{\theta}_i \partial \hat{\theta}_l} \frac{\partial^2 \hat{\Sigma}}{\partial \hat{\theta}_j \partial \hat{\theta}_k} \\
& \left. + (\hat{\Sigma} - \mathbf{S}) \frac{\partial^4 \hat{\Sigma}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k \partial \hat{\theta}_l} \right\}, \tag{A8}
\end{aligned}$$

$$\frac{\partial^4 \hat{F}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k \partial s_{ab}} = -(2 - \delta_{ab}) \frac{\partial^3 \hat{\sigma}_{ab}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k}, \tag{A9}$$

$$(i, j, k, l = 1, \dots, q; p \geq a \geq b \geq 1),$$

where $\hat{\Sigma} - \mathbf{S}$, when evaluated at the true values, becomes $\Sigma(\boldsymbol{\theta}) - \Sigma_{\mathbf{T}} \neq \mathbf{O}$.

2.2 $F_{\rho} = (1/2)\text{tr}\{(\mathbf{P}(\boldsymbol{\theta}) - \mathbf{R})^2\}$

Let $\hat{\rho}_{ab} = (\hat{\mathbf{P}})_{ab} = \{\mathbf{P}(\hat{\boldsymbol{\theta}})\}_{ab}$ and $\hat{\theta}_{\rho i} = (\hat{\boldsymbol{\theta}}_{\rho})_i$, then

$$\frac{\partial^2 \hat{F}_{\rho}}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho j}} = \text{tr} \left\{ \frac{\partial \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho i}} \frac{\partial \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho j}} + (\hat{\mathbf{P}} - \mathbf{R}) \frac{\partial^2 \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho j}} \right\}, \tag{A10}$$

$$\begin{aligned}
\frac{\partial^2 \hat{F}_{\rho}}{\partial \hat{\theta}_{\rho i} \partial s_{ab}} &= -\text{tr} \left(\frac{\partial \mathbf{R}}{\partial s_{ab}} \frac{\partial \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho i}} \right) \\
&= -(2 - \delta_{ab}) s_{aa}^{-1/2} s_{bb}^{-1/2} \frac{\partial \hat{\rho}_{ab}}{\partial \hat{\theta}_{\rho i}} + \delta_{ab} s_{aa}^{-3/2} \sum_{j=1}^p s_{aj} s_{jj}^{-1/2} \frac{\partial \hat{\rho}_{aj}}{\partial \hat{\theta}_{\rho i}}, \tag{A11}
\end{aligned}$$

$$\frac{\partial^3 \hat{F}_\rho}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho j} \partial \hat{\theta}_{\rho k}} = \text{tr} \left\{ \frac{\partial \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho i}} \frac{\partial^2 \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho j} \partial \hat{\theta}_{\rho k}} + \frac{\partial \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho j}} \frac{\partial^2 \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho k}} \right. \\ \left. + \frac{\partial \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho k}} \frac{\partial^2 \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho j}} + (\hat{\mathbf{P}} - \mathbf{R}) \frac{\partial^3 \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho j} \partial \hat{\theta}_{\rho k}} \right\}, \quad (\text{A12})$$

$$\frac{\partial^3 \hat{F}_\rho}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho j} \partial s_{ab}} = -\text{tr} \left(\frac{\partial \mathbf{R}}{\partial s_{ab}} \frac{\partial^2 \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho j}} \right) \\ = -(2 - \delta_{ab}) s_{aa}^{-1/2} s_{bb}^{-1/2} \frac{\partial^2 \hat{\rho}_{ab}}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho j}} \\ + \delta_{ab} s_{aa}^{-3/2} \sum_{k=1}^p s_{ak} s_{kk}^{-1/2} \frac{\partial^2 \hat{\rho}_{ak}}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho j}}, \quad (\text{A13})$$

$$\frac{\partial^3 \hat{F}_\rho}{\partial \hat{\theta}_{\rho i} \partial s_{ab} \partial s_{cd}} = -\text{tr} \left(\frac{\partial^2 \mathbf{R}}{\partial s_{ab} \partial s_{cd}} \frac{\partial \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho i}} \right) \\ = \frac{1}{2} (2 - \delta_{ab}) \delta_{cd} (\delta_{ac} s_{aa}^{-3/2} s_{bb}^{-1/2} + \delta_{bc} s_{aa}^{-1/2} s_{bb}^{-3/2}) \frac{\partial \hat{\rho}_{ab}}{\partial \hat{\theta}_{\rho i}} \\ - \frac{3}{2} \delta_{ab} \delta_{cd} \delta_{ac} s_{aa}^{-5/2} \sum_{j=1}^p s_{aj} s_{jj}^{-1/2} \frac{\partial \hat{\rho}_{aj}}{\partial \hat{\theta}_{\rho i}} \\ + \delta_{ab} s_{aa}^{-3/2} \frac{2 - \delta_{cd}}{2} \left(\delta_{ac} s_{dd}^{-1/2} \frac{\partial \hat{\rho}_{ad}}{\partial \hat{\theta}_{\rho i}} + \delta_{ad} s_{cc}^{-1/2} \frac{\partial \hat{\rho}_{ac}}{\partial \hat{\theta}_{\rho i}} \right) \\ - \frac{1}{2} \delta_{ab} s_{aa}^{-3/2} \delta_{cd} s_{ac} s_{cc}^{-3/2} \frac{\partial \hat{\rho}_{ac}}{\partial \hat{\theta}_{\rho i}}, \quad (\text{A14})$$

$$\begin{aligned}
\frac{\partial^4 \hat{F}_\rho}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho j} \partial \hat{\theta}_{\rho k} \partial \hat{\theta}_{\rho l}} &= \text{tr} \left\{ \frac{\partial \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho i}} \frac{\partial^3 \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho j} \partial \hat{\theta}_{\rho k} \partial \hat{\theta}_{\rho l}} \right. \\
&+ \frac{\partial \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho j}} \frac{\partial^3 \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho k} \partial \hat{\theta}_{\rho l}} + \frac{\partial \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho k}} \frac{\partial^3 \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho j} \partial \hat{\theta}_{\rho l}} \\
&+ \frac{\partial \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho l}} \frac{\partial^3 \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho j} \partial \hat{\theta}_{\rho k}} + \frac{\partial^2 \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho j}} \frac{\partial^2 \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho k} \partial \hat{\theta}_{\rho l}} \\
&+ \frac{\partial^2 \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho k}} \frac{\partial^2 \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho j} \partial \hat{\theta}_{\rho l}} + \frac{\partial^2 \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho l}} \frac{\partial^2 \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho j} \partial \hat{\theta}_{\rho k}} \\
&\left. + (\hat{\mathbf{P}} - \mathbf{R}) \frac{\partial^4 \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho j} \partial \hat{\theta}_{\rho k} \partial \hat{\theta}_{\rho l}} \right\}, \tag{A15}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^4 \hat{F}_\rho}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho j} \partial \hat{\theta}_{\rho k} \partial s_{ab}} &= -\text{tr} \left(\frac{\partial \mathbf{R}}{\partial s_{ab}} \frac{\partial^3 \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho j} \partial \hat{\theta}_{\rho k}} \right) \\
&= -(2 - \delta_{ab}) s_{aa}^{-1/2} s_{bb}^{-1/2} \frac{\partial^3 \hat{\rho}_{ab}}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho j} \partial \hat{\theta}_{\rho k}} \\
&+ \delta_{ab} s_{aa}^{-3/2} \sum_{l=1}^p s_{al} s_{ll}^{-1/2} \frac{\partial^2 \hat{\rho}_{al}}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho j} \partial \hat{\theta}_{\rho k}}, \tag{A16}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^4 \hat{F}_\rho}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho j} \partial s_{ab} \partial s_{cd}} &= \frac{1}{2} (2 - \delta_{ab}) \delta_{cd} (\delta_{ac} s_{aa}^{-3/2} s_{bb}^{-1/2} + \delta_{bc} s_{aa}^{-1/2} s_{bb}^{-3/2}) \\
&\times \frac{\partial^2 \hat{\rho}_{ab}}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho j}} - \frac{3}{2} \delta_{ab} \delta_{cd} \delta_{ac} s_{aa}^{-5/2} \sum_{k=1}^p s_{ak} s_{kk}^{-1/2} \frac{\partial^2 \hat{\rho}_{ak}}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho j}} \\
&+ \delta_{ab} s_{aa}^{-3/2} \frac{2 - \delta_{cd}}{2} \left(\delta_{ac} s_{dd}^{-1/2} \frac{\partial^2 \hat{\rho}_{ad}}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho j}} + \delta_{ad} s_{cc}^{-1/2} \frac{\partial^2 \hat{\rho}_{ac}}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho j}} \right) \quad (\text{A17}) \\
&- \frac{1}{2} \delta_{ab} s_{aa}^{-3/2} \delta_{cd} s_{ac} s_{cc}^{-3/2} \frac{\partial^2 \hat{\rho}_{ac}}{\partial \hat{\theta}_{\rho i} \partial \hat{\theta}_{\rho j}},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^4 \hat{F}_\rho}{\partial \hat{\theta}_{\rho i} \partial s_{ab} \partial s_{cd} \partial s_{ef}} &= -\text{tr} \left(\frac{\partial^3 \mathbf{R}}{\partial s_{ab} \partial s_{cd} \partial s_{ef}} \frac{\partial \hat{\mathbf{P}}}{\partial \hat{\theta}_{\rho i}} \right) \\
&= -\frac{1}{4} (2 - \delta_{ab}) \delta_{cd} (3 \delta_{ac} \delta_{ef} \delta_{ea} s_{aa}^{-5/2} s_{bb}^{-1/2} + \delta_{ac} \delta_{ef} \delta_{eb} s_{aa}^{-3/2} s_{bb}^{-3/2} \\
&\quad + \delta_{bc} \delta_{ef} \delta_{ea} s_{aa}^{-3/2} s_{bb}^{-3/2} + 3 \delta_{bc} \delta_{ef} \delta_{eb} s_{aa}^{-1/2} s_{bb}^{-5/2}) \frac{\partial \hat{\rho}_{ab}}{\partial \hat{\theta}_{\rho i}} \\
&+ \frac{15}{4} \delta_{ab} \delta_{cd} \delta_{ac} \delta_{ef} \delta_{ae} s_{aa}^{-7/2} \sum_{j=1}^p s_{aj} s_{jj}^{-1/2} \frac{\partial \hat{\rho}_{aj}}{\partial \hat{\theta}_{\rho i}} - \frac{3}{2} \delta_{ab} \delta_{cd} \delta_{ac} s_{aa}^{-5/2} \\
&\quad \times \left\{ \frac{2 - \delta_{ef}}{2} \left(\delta_{ae} s_{ff}^{-1/2} \frac{\partial \hat{\rho}_{af}}{\partial \hat{\theta}_{\rho i}} + \delta_{af} s_{ee}^{-1/2} \frac{\partial \hat{\rho}_{ae}}{\partial \hat{\theta}_{\rho i}} \right) - \frac{1}{2} \delta_{ef} s_{ae} s_{ee}^{-3/2} \frac{\partial \hat{\rho}_{ae}}{\partial \hat{\theta}_{\rho i}} \right\} \\
&- \frac{3}{4} \delta_{ab} \delta_{ef} \delta_{ae} s_{aa}^{-5/2} (2 - \delta_{cd}) \left(\delta_{ac} s_{dd}^{-1/2} \frac{\partial \hat{\rho}_{ad}}{\partial \hat{\theta}_{\rho i}} + \delta_{ad} s_{cc}^{-1/2} \frac{\partial \hat{\rho}_{ac}}{\partial \hat{\theta}_{\rho i}} \right) \\
&- \frac{1}{4} \delta_{ab} s_{aa}^{-3/2} (2 - \delta_{cd}) \delta_{ef} \left(\delta_{ac} \delta_{de} s_{dd}^{-3/2} \frac{\partial \hat{\rho}_{ad}}{\partial \hat{\theta}_{\rho i}} + \delta_{ad} \delta_{ce} s_{cc}^{-3/2} \frac{\partial \hat{\rho}_{ac}}{\partial \hat{\theta}_{\rho i}} \right) \\
&+ \frac{1}{2} \delta_{ab} \delta_{cd} \left\{ \frac{3}{2} \delta_{ef} \delta_{ea} s_{aa}^{-5/2} s_{ac} s_{cc}^{-3/2} - s_{aa}^{-3/2} \frac{2 - \delta_{ef}}{2} (\delta_{ae} \delta_{cf} + \delta_{af} \delta_{ce}) s_{cc}^{-3/2} \right. \\
&\quad \left. + \frac{3}{2} \delta_{ef} \delta_{ec} s_{aa}^{-3/2} s_{ac} s_{cc}^{-5/2} \right\} \frac{\partial \hat{\rho}_{ac}}{\partial \hat{\theta}_{\rho i}},
\end{aligned}$$

$$(i, j, k, l = 1, \dots, q; p \geq a \geq b \geq 1; p \geq c \geq d \geq 1; p \geq e \geq f \geq 1),$$

(A18)

where $\hat{\mathbf{P}} - \mathbf{R}$, when evaluated at the true values, is $\mathbf{P}(\boldsymbol{\theta}_\rho) - \mathbf{P}_T \neq \mathbf{O}$. In the above results, s_{aa} 's and s_{ab} 's ($a, b = 1, \dots, p; a \neq b$) can be replaced by 1 and r_{ab} 's, respectively without loss of generality.

References

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