Supplement to the paper "Asymptotic properties of the Bayes modal estimators of item parameters in item response theory"

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This note is to supplement Ogasawara (2013), and gives asymptotic expansions for the reciprocals of the estimated asymptotic standard errors used for studentized estimators and a simplification of Theorem 6.

(a) t

Define $\hat{i}_{\mathrm{W}}^{\alpha\alpha} = (\hat{\mathbf{I}}_{\mathrm{W}}^{-1})_{(\alpha\alpha)}$. Then, for $t = N^{1/2}(\hat{\alpha}_{\mathrm{W}} - \alpha_0) / (\hat{i}_{\mathrm{W}}^{\alpha\alpha})^{1/2}$,

$$(\hat{i}_{\mathrm{W}}^{\alpha\alpha})^{-1/2} = (i_0^{\alpha\alpha})^{-1/2} + \frac{1}{2}(i_0^{\alpha\alpha})^{-3/2} \sum_{a \geq b} \frac{2 - \delta_{ab}}{2} \{\mathbf{I}_0^{-1}(\mathbf{E}_{ab} + \mathbf{E}_{ba})\mathbf{I}_0^{-1}\}_{(\alpha\alpha)} \frac{\partial i_{0ab}}{\partial \boldsymbol{\alpha}_0}'(\hat{\boldsymbol{\alpha}}_{\mathrm{W}} - \boldsymbol{\alpha}_0)$$

$$+ \frac{3}{8}(i_0^{\alpha\alpha})^{-5/2} \left[\sum_{a \geq b} \frac{2 - \delta_{ab}}{2} \{\mathbf{I}_0^{-1}(\mathbf{E}_{ab} + \mathbf{E}_{ba})\mathbf{I}_0^{-1}\}_{(\alpha\alpha)} \frac{\partial i_{0ab}}{\partial \boldsymbol{\alpha}_0}'(\hat{\boldsymbol{\alpha}}_{\mathrm{W}} - \boldsymbol{\alpha}_0) \right]^2$$

$$- \frac{1}{4}(i_0^{\alpha\alpha})^{-3/2} \sum_{a \geq b} \sum_{c \geq d} \frac{1}{4}(2 - \delta_{ab})(2 - \delta_{cd})$$

$$\times 2 \{\mathbf{I}_0^{-1}(\mathbf{E}_{ab} + \mathbf{E}_{ba})\mathbf{I}_0^{-1}(\mathbf{E}_{cd} + \mathbf{E}_{dc})\mathbf{I}_0^{-1}\}_{(\alpha\alpha)} \frac{\partial i_{0ab}}{\partial \boldsymbol{\alpha}_0}'(\hat{\boldsymbol{\alpha}}_{\mathrm{W}} - \boldsymbol{\alpha}_0) \frac{\partial i_{0cd}}{\partial \boldsymbol{\alpha}_0}'(\hat{\boldsymbol{\alpha}}_{\mathrm{W}} - \boldsymbol{\alpha}_0)$$

$$+ \frac{1}{4}(i_0^{\alpha\alpha})^{-3/2} \sum_{a \geq b} \frac{2 - \delta_{ab}}{2} \{\mathbf{I}_0^{-1}(\mathbf{E}_{ab} + \mathbf{E}_{ba})\mathbf{I}_0^{-1}\}_{(\alpha\alpha)} \frac{\partial^2 i_{0ab}}{(\partial \boldsymbol{\alpha}_0')^{<2>}}(\hat{\boldsymbol{\alpha}}_{\mathrm{W}} - \boldsymbol{\alpha}_0)^{<2>}$$

$$+ O_p(N^{-3/2})$$

$$\equiv (i_0^{\alpha\alpha})^{-1/2} + \mathbf{i}_0^{(1)}{}'(\hat{\boldsymbol{\alpha}}_{\mathrm{W}} - \boldsymbol{\alpha}_0) + \mathbf{i}_0^{(2)}{}'(\hat{\boldsymbol{\alpha}}_{\mathrm{W}} - \boldsymbol{\alpha}_0)^{<2>} + O_p(N^{-3/2})$$

$$= (i_0^{\alpha\alpha})^{-1/2} + \mathbf{i}_0^{(1)}{}'(N^{-1}\boldsymbol{\eta}_0 + N^{-1/2}\boldsymbol{\Lambda}^{(1)}\mathbf{m}^{(1)} + N^{-1}\boldsymbol{\Lambda}^{(2)}\mathbf{m}^{(2)})$$

$$+ \mathbf{i}_0^{(2)}{}'(N^{-1/2}\boldsymbol{\Lambda}^{(1)}\mathbf{m}^{(1)})^{<2>} + O_p(N^{-3/2}) \tag{S.1}$$

$$= (i_0^{;\alpha\alpha})^{-1/2} + N^{-1/2}\mathbf{i}_0^{(1)}{}'\mathbf{\Lambda}^{(1)}\mathbf{m}^{(1)} + N^{-1}\mathbf{i}_0^{(1)}{}'\mathbf{\eta}_0$$
$$+ N^{-1}\{\mathbf{i}_0^{(1)}{}'\mathbf{\Lambda}^{(2)}\mathbf{m}^{(2)} + \mathbf{i}_0^{(2)}{}'(\mathbf{\Lambda}^{(1)}\mathbf{m}^{(1)})^{<2>}\} + O_p(N^{-3/2})$$

$$\equiv (i_0^{;\alpha\alpha})^{-1/2} + N^{-1}\eta_{t0} + \sum_{j=1}^{2} N^{-j/2}\mathbf{\lambda}_t^{(j)}{}'\mathbf{m}^{(j)} + O_p(N^{-3/2}),$$

where $\displaystyle\sum_{a\geq b}(\cdot) = \sum_{a=1}^{q}\sum_{b=1}^{a}(\cdot)$, δ_{ab} is the Kronecker delta, $\mathbf{E}_{ab}$ is a square matrix of an appropriate size whose (a, b)th element is 1 with other ones being zero, and $\mathbf{\Lambda}^{(j)}(j=1,2)$ are multivariate versions of $\mathbf{\lambda}^{(j)}{}'(j=1,2)$, respectively (see (8.2)).

In (S.1),
$$\mathbf{i}_0^{(1)} = \frac{1}{2}(i_0^{;\alpha\alpha})^{-3/2}\sum_{a\geq b}(2-\delta_{ab})i_0^{;\alpha a}i_0^{;\alpha b}\frac{\partial i_{0ab}}{\partial\mathbf{\alpha}_0},$$

$$\mathbf{i}_0^{(2)} = \frac{3}{8}(i_0^{;\alpha\alpha})^{-5/2}\left[\sum_{a\geq b}(2-\delta_{ab})i_0^{;\alpha a}i_0^{;\alpha b}\frac{\partial i_{0ab}}{\partial\mathbf{\alpha}_0}\right]^{<2>}$$
$$-\frac{1}{8}(i_0^{;\alpha\alpha})^{-3/2}\sum_{a\geq b}\sum_{c\geq d}(2-\delta_{ab})(2-\delta_{cd})\sum_{(abcd)}^{4}i_0^{;\alpha a}i_0^{;bc}i_0^{;d\alpha}\frac{\partial i_{0ab}}{\partial\mathbf{\alpha}_0}\otimes\frac{\partial i_{0cd}}{\partial\mathbf{\alpha}_0} \quad \text{(S.2)}$$
$$+\frac{1}{4}(i_0^{;\alpha\alpha})^{-3/2}\sum_{a\geq b}(2-\delta_{ab})i_0^{;\alpha a}i_0^{;\alpha b}\frac{\partial^2 i_{0ab}}{(\partial\mathbf{\alpha}_0)^{<2>}},$$

$\eta_{t0} = \mathbf{i}_0^{(1)}{}'\mathbf{\eta}_0$, $\mathbf{\lambda}_t^{(1)}{}' = \mathbf{i}_0^{(1)}{}'\mathbf{\Lambda}^{(1)}$, $\mathbf{\lambda}_t^{(2)}{}' = \mathbf{i}_0^{(1)}{}'\mathbf{\Lambda}^{(2)} + \mathbf{i}_0^{(2)}{}'\mathbf{\Lambda}^{(1)<2>}$, where $\displaystyle\sum_{(abcd)}^{4}$ is the sum of four terms considering the symmetric cases for a, b, c

and d, and noting $\displaystyle\mathbf{I}_0 \equiv \sum_{k=1}^{K}\frac{1}{\pi_{0k}}\frac{\partial\pi_{0k}}{\partial\mathbf{\alpha}_0}\frac{\partial\pi_{0k}}{\partial\mathbf{\alpha}_0}{}'$ with $\pi_{0k} \equiv (\mathbf{\pi}_0)_k \equiv \{\mathbf{\pi}(\mathbf{\alpha}_0)\}_k$, we have

$$\frac{\partial i_{0ab}}{\partial \alpha_{0c}} = \sum_{k=1}^{K}\left(-\frac{1}{\pi_{0k}^2}\frac{\partial \pi_{0k}}{\partial \alpha_{0a}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}}\frac{\partial \pi_{0k}}{\partial \alpha_{0c}}+\sum_{(ab)}^{2}\frac{1}{\pi_{0k}}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0c}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}}\right),$$

$$\frac{\partial^2 i_{0ab}}{\partial \alpha_{0c}\partial \alpha_{0d}} = \sum_{k=1}^{K}\left\{\frac{2}{\pi_{0k}^3}\frac{\partial \pi_{0k}}{\partial \alpha_{0a}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}}\frac{\partial \pi_{0k}}{\partial \alpha_{0c}}\frac{\partial \pi_{0k}}{\partial \alpha_{0d}}\right.$$

$$-\frac{1}{\pi_{0k}^2}\left(\sum_{(ab)}^{2}\sum_{(cd)}^{2}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0c}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}}\frac{\partial \pi_{0k}}{\partial \alpha_{0d}}+\frac{\partial \pi_{0k}}{\partial \alpha_{0a}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0c}\partial \alpha_{0d}}\right)$$

$$+\frac{1}{\pi_{0k}}\left(\sum_{(ab)}^{2}\frac{\partial^3 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0c}\partial \alpha_{0d}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}}+\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0c}}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0b}\partial \alpha_{0d}}\right.$$

$$\left.\left.+\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0d}}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0b}\partial \alpha_{0c}}\right)\right\}$$

(S.3)

$(a,b,c,d = 1,...,q).$

(b) t_H

Using $\hat{\mathbf{H}}_{\mathrm{W}}$ (see (4.2b)), define $\hat{h}_{\mathrm{W}}^{\alpha\alpha}$, $\mathbf{H}_0$, $\hat{\mathbf{D}}_{\mathrm{W}}$, $\hat{d}_{\mathrm{W}}^{\alpha\alpha}$ and $d_0^{\alpha\alpha}$ in the following equations:

$$t_H \equiv \frac{N^{1/2}(\hat{\alpha}_{\mathrm{W}}-\alpha_0)}{\{(-\hat{\mathbf{H}}_{\mathrm{W}}^{-1})_{(\alpha\alpha)}\}^{1/2}} \equiv \frac{N^{1/2}(\hat{\alpha}_{\mathrm{W}}-\alpha_0)}{(-\hat{h}_{\mathrm{W}}^{\alpha\alpha})^{1/2}},$$

$$-\hat{h}_{\mathrm{W}}^{\alpha\alpha} = \left[\left\{\sum_{k=1}^{K}p_k\left(-\frac{1}{\hat{\pi}_k}\frac{\partial^2 \hat{\pi}_k}{\partial \hat{\boldsymbol{\alpha}}_{\mathrm{W}}\partial \hat{\boldsymbol{\alpha}}_{\mathrm{W}}'}+\frac{1}{\hat{\pi}_k^2}\frac{\partial \hat{\pi}_k}{\partial \hat{\boldsymbol{\alpha}}_{\mathrm{W}}}\frac{\partial \hat{\pi}_k}{\partial \hat{\boldsymbol{\alpha}}_{\mathrm{W}}'}\right)\right\}^{-1}\right]_{(\alpha\alpha)},$$

(S.4)

$$\hat{\mathbf{D}}_{\mathrm{W}} = -\hat{\mathbf{H}}_{\mathrm{W}}, \ \mathbf{D}_0 = -\mathbf{H}_0 = \sum_{k=1}^{K}\pi_{\mathrm{T}k}\left(-\frac{1}{\pi_{0k}}\frac{\partial^2 \pi_{0k}}{\partial \boldsymbol{\alpha}_0\partial \boldsymbol{\alpha}_0'}+\frac{1}{\pi_{0k}^2}\frac{\partial \pi_{0k}}{\partial \boldsymbol{\alpha}_0}\frac{\partial \pi_{0k}}{\partial \boldsymbol{\alpha}_0'}\right).$$

Then,

$$(-\hat{h}_{\mathrm{W}}^{\alpha\alpha})^{-1/2} = (\hat{d}_{\mathrm{W}}^{\alpha\alpha})^{-1/2} = (d_0^{\alpha\alpha})^{-1/2}$$

$$+\frac{1}{2}(d_0^{\alpha\alpha})^{-3/2}\sum_{a\geq b}(2-\delta_{ab})d_0^{\alpha a}d_0^{\alpha b}\frac{\partial d_{0ab}}{\partial(\boldsymbol{\pi}_{\mathrm{T}}',\boldsymbol{\alpha}_0')}\{(\mathbf{p}-\boldsymbol{\pi}_{\mathrm{T}})',(\hat{\boldsymbol{\alpha}}_{\mathrm{W}}-\boldsymbol{\alpha}_0)'\}'$$

(S.5)

$$+\left[\frac{3}{8}(d_0^{\alpha\alpha})^{-5/2}\left\{\sum_{a\geq b}(2-\delta_{ab})d_0^{\alpha a}d_0^{\alpha b}\frac{\partial d_{0ab}}{\partial(\boldsymbol{\pi}_T\,',\boldsymbol{\alpha}_0\,')}\right\}^{<2>}\right.$$

$$-\frac{1}{8}(d_0^{\alpha\alpha})^{-3/2}\sum_{a\geq b}\sum_{c\geq d^*}(2-\delta_{ab})(2-\delta_{cd^*})$$

$$\times\sum_{(abcd^*)}^{4}d_0^{\alpha a}d_0^{bc}d_0^{d^*\alpha}\frac{\partial d_{0ab}}{\partial(\boldsymbol{\pi}_T\,',\boldsymbol{\alpha}_0\,')}\otimes\frac{\partial d_{0cd^*}}{\partial(\boldsymbol{\pi}_T\,',\boldsymbol{\alpha}_0\,')}$$

$$\left.+\frac{1}{4}(d_0^{\alpha\alpha})^{-3/2}\sum_{a\geq b}(2-\delta_{ab})d_0^{\alpha a}d_0^{\alpha b}\frac{\partial^2 d_{0ab}}{\{\partial(\boldsymbol{\pi}_T\,',\boldsymbol{\alpha}_0\,')\}^{<2>}}\right]\{(\mathbf{p}-\boldsymbol{\pi}_T)',(\hat{\boldsymbol{\alpha}}_W-\boldsymbol{\alpha}_0)'\}^{<2>}$$

$$\equiv(d_0^{\alpha\alpha})^{-1/2}+\mathbf{d}_0^{(1)}\,'\{(\mathbf{p}-\boldsymbol{\pi}_T)',(\hat{\boldsymbol{\alpha}}_W-\boldsymbol{\alpha}_0)'\}'$$

$$+\mathbf{d}_0^{(2)}\,'\{(\mathbf{p}-\boldsymbol{\pi}_T)',(\hat{\boldsymbol{\alpha}}_W-\boldsymbol{\alpha}_0)'\}^{'<2>}+O_p(N^{-3/2})$$

$$=(d_0^{\alpha\alpha})^{-1/2}+\mathbf{d}_0^{(1)}\,'\{N^{-1/2}\mathbf{m}^{(1)}\,',(N^{-1}\boldsymbol{\eta}_0+N^{-1/2}\boldsymbol{\Lambda}^{(1)}\mathbf{m}^{(1)}+N^{-1}\boldsymbol{\Lambda}^{(2)}\mathbf{m}^{(2)})'\}'$$

$$+\mathbf{d}_0^{(2)}\,'\{N^{-1/2}\mathbf{m}^{(1)}\,',N^{-1/2}(\boldsymbol{\Lambda}^{(1)}\mathbf{m}^{(1)})'\}^{'<2>}+O_p(N^{-3/2}).$$

Decomposing $\mathbf{d}_0^{(j)}(j=1,2)$ using subvectors as

$$\mathbf{d}_0^{(1)}=(\mathbf{d}_0^{(1A)}\,',\mathbf{d}_0^{(1B)}\,')'\quad\text{and}\quad\mathbf{d}_0^{(2)}=(\mathbf{d}_0^{(2AA)}\,',\mathbf{d}_0^{(2AB)}\,',\mathbf{d}_0^{(2BA)}\,',\mathbf{d}_0^{(2BB)}\,')',\qquad\text{(S.6)}$$

(S.5) becomes

$$=(d_0^{\alpha\alpha})^{-1/2}+N^{-1/2}(\mathbf{d}_0^{(1A)}\,'+\mathbf{d}_0^{(1B)}\,'\boldsymbol{\Lambda}^{(1)})\mathbf{m}^{(1)}+N^{-1}\mathbf{d}_0^{(1B)}\,'\boldsymbol{\eta}_0$$

$$+N^{-1}\{\mathbf{d}_0^{(1B)}\,'\boldsymbol{\Lambda}^{(2)}+\mathbf{d}_0^{(2AA)}\,'+2\mathbf{d}_0^{(2AB)}\,'(\mathbf{I}_{(K)}\otimes\boldsymbol{\Lambda}^{(1)})+\mathbf{d}_0^{(2BB)}\,'\boldsymbol{\Lambda}^{(1)<2>}\}\mathbf{m}^{(1)<2>}$$

$$+O_p(N^{-3/2})$$

$$\equiv(d_0^{\alpha\alpha})^{-1/2}+N^{-1}\eta_{H0}+\sum_{j=1}^{2}N^{-j/2}\boldsymbol{\lambda}_H^{(j)}\,'\mathbf{m}^{(j)}+O_p(N^{-3/2}),$$

(S.7)

where

$$\eta_{H0}=\mathbf{d}_0^{(1B)}\,'\boldsymbol{\eta}_0,\quad\boldsymbol{\lambda}_H^{(1)}\,'=\mathbf{d}_0^{(1A)}\,'+\mathbf{d}_0^{(1B)}\,'\boldsymbol{\Lambda}^{(1)},$$

$$\boldsymbol{\lambda}_H^{(2)}\,'=\mathbf{d}_0^{(1B)}\,'\boldsymbol{\Lambda}^{(2)}+\mathbf{d}_0^{(2AA)}\,'+2\mathbf{d}_0^{(2AB)}\,'(\mathbf{I}_{(K)}\otimes\boldsymbol{\Lambda}^{(1)})+\mathbf{d}_0^{(2BB)}\,'\boldsymbol{\Lambda}^{(1)<2>}\qquad\text{(S.8)}$$

and $\mathbf{I}_{(K)}$ is the $K\times K$ identity matrix.

In (S.5),

$$\frac{\partial d_{0ab}}{\partial \pi_{\mathrm{T}k*}} = -\frac{1}{\pi_{0k*}}\frac{\partial^2 \pi_{0k*}}{\partial \alpha_{0a}\partial \alpha_{0b}} + \frac{1}{\pi_{0k*}^2}\frac{\partial \pi_{0k*}}{\partial \alpha_{0a}}\frac{\partial \pi_{0k*}}{\partial \alpha_{0b}},$$

$$\frac{\partial d_{0ab}}{\partial \alpha_{0c}} = \sum_{k=1}^{K}\pi_{\mathrm{T}k}\left(-\frac{2}{\pi_{0k}^3}\frac{\partial \pi_{0k}}{\partial \alpha_{0a}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}}\frac{\partial \pi_{0k}}{\partial \alpha_{0c}} + \frac{1}{\pi_{0k}^2}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0b}}\frac{\partial \pi_{0k}}{\partial \alpha_{0c}}\right.$$

$$\left. + \sum_{(ab)}^{2}\frac{1}{\pi_{0k}^2}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0c}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}} - \frac{1}{\pi_{0k}}\frac{\partial^3 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0b}\partial \alpha_{0c}}\right),$$

$$\frac{\partial^2 d_{0ab}}{\partial \pi_{\mathrm{T}k*}\partial \pi_{\mathrm{T}l}} = 0,$$

$$\frac{\partial^2 d_{0ab}}{\partial \pi_{\mathrm{T}k*}\partial \alpha_{0c}} = -\frac{2}{\pi_{0k*}^3}\frac{\partial \pi_{0k*}}{\partial \alpha_{0a}}\frac{\partial \pi_{0k*}}{\partial \alpha_{0b}}\frac{\partial \pi_{0k*}}{\partial \alpha_{0c}} + \frac{1}{\pi_{0k*}^2}\frac{\partial^2 \pi_{0k*}}{\partial \alpha_{0a}\partial \alpha_{0b}}\frac{\partial \pi_{0k*}}{\partial \alpha_{0c}}$$

$$+ \frac{1}{\pi_{0k*}^2}\sum_{(ab)}^{2}\frac{\partial^2 \pi_{0k*}}{\partial \alpha_{0a}\partial \alpha_{0c}}\frac{\partial \pi_{0k*}}{\partial \alpha_{0b}} - \frac{1}{\pi_{0k*}}\frac{\partial^3 \pi_{0k*}}{\partial \alpha_{0a}\partial \alpha_{0b}\partial \alpha_{0c}},$$

$$\frac{\partial^2 d_{0ab}}{\partial \alpha_{0c}\partial \alpha_{0d*}} = \sum_{k=1}^{K}\pi_{\mathrm{T}k}\left\{\frac{6}{\pi_{0k}^4}\frac{\partial \pi_{0k}}{\partial \alpha_{0a}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}}\frac{\partial \pi_{0k}}{\partial \alpha_{0c}}\frac{\partial \pi_{0k}}{\partial \alpha_{0d*}}\right.$$

$$-\frac{2}{\pi_{0k}^3}\left(\sum_{(abcd*)}^{4}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0c}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}}\frac{\partial \pi_{0k}}{\partial \alpha_{0d*}} + \frac{\partial \pi_{0k}}{\partial \alpha_{0a}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0c}\partial \alpha_{0d*}}\right.\tag{S.9}$$

$$\left.+ \frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0b}}\frac{\partial \pi_{0k}}{\partial \alpha_{0c}}\frac{\partial \pi_{0k}}{\partial \alpha_{0d*}}\right)$$

$$+ \frac{1}{\pi_{0k}^2}\left(\sum_{(abcd*)}^{4}\frac{\partial^3 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0b}\partial \alpha_{0c}}\frac{\partial \pi_{0k}}{\partial \alpha_{0d*}} + \frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0b}}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0c}\partial \alpha_{0d*}}\right.$$

$$\left.+ \frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0c}}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0b}\partial \alpha_{0d*}} + \frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0d*}}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0b}\partial \alpha_{0c}}\right)$$

$$\left.-\frac{1}{\pi_{0k}}\frac{\partial^4 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0b}\partial \alpha_{0c}\partial \alpha_{0d*}}\right\}$$

$$(a,b,c,d^* = 1,...,q; k^*,l = 1,...,K).$$

(c) t_G

Using $\hat{\mathbf{G}}_W$ (see (4.2c)), define $\hat{g}_W^{\alpha\alpha}$, $\mathbf{G}_0$ and $g_0^{\alpha\alpha}$ as

$$t_G \equiv \frac{N^{1/2}(\hat{\alpha}_W - \alpha_0)}{\{(\hat{\mathbf{G}}_W^{-1})_{(\alpha\alpha)}\}^{1/2}} \equiv \frac{N^{1/2}(\hat{\alpha}_W - \alpha_0)}{(\hat{g}_W^{\alpha\alpha})^{1/2}},$$

$$\mathbf{G}_0 = \sum_{k=1}^{K} \frac{\pi_{Tk}}{\pi_{0k}^2} \frac{\partial \pi_{0k}}{\partial \boldsymbol{\alpha}_0} \frac{\partial \pi_{0k}}{\partial \boldsymbol{\alpha}_0{}'} \qquad \text{(S.10)}$$

and $g_0^{\alpha\alpha} = (\mathbf{G}_0^{-1})_{(\alpha\alpha)}$.

Then,

$$(\hat{g}_W^{\alpha\alpha})^{-1/2} = (g_0^{\alpha\alpha})^{-1/2}$$

$$+ \frac{1}{2}(g_0^{\alpha\alpha})^{-3/2} \sum_{a \ge b}(2-\delta_{ab}) g_0^{\alpha a} g_0^{\alpha b} \frac{\partial g_{0ab}}{\partial(\boldsymbol{\pi}_T{}', \boldsymbol{\alpha}_0{}')} \{(\mathbf{p}-\boldsymbol{\pi}_T)', (\hat{\boldsymbol{\alpha}}_W - \boldsymbol{\alpha}_0)'\}' \quad \text{(S.11)}$$

$$+ \left[\frac{3}{8}(g_0^{\alpha\alpha})^{-5/2} \left\{ \sum_{a \ge b}(2-\delta_{ab}) g_0^{\alpha a} g_0^{\alpha b} \frac{\partial g_{0ab}}{\partial(\boldsymbol{\pi}_T{}', \boldsymbol{\alpha}_0{}')} \right\}^{<2>} \right.$$

$$- \frac{1}{8}(g_0^{\alpha\alpha})^{-3/2} \sum_{a \ge b} \sum_{c \ge d*}(2-\delta_{ab})(2-\delta_{cd*})$$

$$\times \sum_{(abcd*)}^{4} g_0^{\alpha a} g_0^{bc} g_0^{d*\alpha} \frac{\partial g_{0ab}}{\partial(\boldsymbol{\pi}_T{}', \boldsymbol{\alpha}_0{}')} \otimes \frac{\partial g_{0cd*}}{\partial(\boldsymbol{\pi}_T{}', \boldsymbol{\alpha}_0{}')}$$

$$\left. + \frac{1}{4}(g_0^{\alpha\alpha})^{-3/2} \sum_{a \ge b}(2-\delta_{ab}) g_0^{\alpha a} g_0^{\alpha b} \frac{\partial^2 g_{0ab}}{\{\partial(\boldsymbol{\pi}_T{}', \boldsymbol{\alpha}_0{}')\}^{<2>}} \right] \{(\mathbf{p}-\boldsymbol{\pi}_T)', (\hat{\boldsymbol{\alpha}}_W - \boldsymbol{\alpha}_0)'\}'^{<2>}$$

$$\equiv (g_0^{\alpha\alpha})^{-1/2} + \mathbf{g}_0^{(1)}{}' \{(\mathbf{p}-\boldsymbol{\pi}_T)', (\hat{\boldsymbol{\alpha}}_W - \boldsymbol{\alpha}_0)'\}'$$

$$+ \mathbf{g}_0^{(2)}{}' \{(\mathbf{p}-\boldsymbol{\pi}_T)', (\hat{\boldsymbol{\alpha}}_W - \boldsymbol{\alpha}_0)'\}'^{<2>} + O_p(N^{-3/2})$$

$$= (g_0^{\alpha\alpha})^{-1/2} + \mathbf{g}_0^{(1)}{}' \{N^{-1/2}\mathbf{m}^{(1)}{}', (N^{-1}\boldsymbol{\eta}_0 + N^{-1/2}\boldsymbol{\Lambda}^{(1)}\mathbf{m}^{(1)} + N^{-1}\boldsymbol{\Lambda}^{(2)}\mathbf{m}^{(2)})'\}'$$

$$+ \mathbf{g}_0^{(2)}{}' \{N^{-1/2}\mathbf{m}^{(1)}{}', N^{-1/2}(\boldsymbol{\Lambda}^{(1)}\mathbf{m}^{(1)})'\}'^{<2>} + O_p(N^{-3/2}).$$

Decomposing $\mathbf{g}_0^{(j)} (j=1,2)$ using subvectors as

$$\mathbf{g}_0^{(1)} = (\mathbf{g}_0^{(1A)}{}', \mathbf{g}_0^{(1B)}{}')' \quad \text{and} \quad \mathbf{g}_0^{(2)} = (\mathbf{g}_0^{(2AA)}{}', \mathbf{g}_0^{(2AB)}{}', \mathbf{g}_0^{(2BA)}{}', \mathbf{g}_0^{(2BB)}{}')', \quad \text{(S.11)}$$

becomes

$$= (g_0^{\alpha\alpha})^{-1/2} + N^{-1/2}(\mathbf{g}_0^{(1A)}{}' + \mathbf{g}_0^{(1B)}{}'\boldsymbol{\Lambda}^{(1)})\mathbf{m}^{(1)} + N^{-1}\mathbf{g}_0^{(1B)}{}'\boldsymbol{\eta}_0$$

$$+ N^{-1}\{\mathbf{g}_0^{(1B)}{}'\boldsymbol{\Lambda}^{(2)} + \mathbf{g}_0^{(2AA)}{}' + 2\mathbf{g}_0^{(2AB)}{}'(\mathbf{I}_{(K)} \otimes \boldsymbol{\Lambda}^{(1)}) + \mathbf{g}_0^{(2BB)}{}'\boldsymbol{\Lambda}^{(1)<2>}\}\mathbf{m}^{(1)<2>}$$

$$+ O_p(N^{-3/2})$$

$$\text{(S.12)}$$

$$\equiv (g_0^{\alpha\alpha})^{-1/2} + N^{-1}\eta_{G0} + \sum_{j=1}^{2} N^{-j/2}\boldsymbol{\lambda}_G^{(j)}{}'\mathbf{m}^{(j)} + O_p(N^{-3/2}),$$

where

$$\eta_{G0} = \mathbf{g}_0^{(1B)}{}'\boldsymbol{\eta}_0, \ \ \boldsymbol{\lambda}_G^{(1)}{}' = \mathbf{g}_0^{(1A)}{}' + \mathbf{g}_0^{(1B)}{}'\boldsymbol{\Lambda}^{(1)},$$

$$\boldsymbol{\lambda}_G^{(2)}{}' = \mathbf{g}_0^{(1B)}{}'\boldsymbol{\Lambda}^{(2)} + \mathbf{g}_0^{(2AA)}{}' + 2\mathbf{g}_0^{(2AB)}{}'(\mathbf{I}_{(K)} \otimes \boldsymbol{\Lambda}^{(1)}) + \mathbf{g}_0^{(2BB)}{}'\boldsymbol{\Lambda}^{(1)<2>} \qquad \text{(S.13)}$$

In (S.11),

$$\frac{\partial g_{0ab}}{\partial \pi_{\mathrm{T}k*}} = \frac{1}{\pi_{0k*}^2}\frac{\partial \pi_{0k*}}{\partial \alpha_{0a}}\frac{\partial \pi_{0k*}}{\partial \alpha_{0b}},$$

$$\frac{\partial g_{0ab}}{\partial \alpha_{0c}} = \sum_{k=1}^{K} \pi_{\mathrm{T}k}\left(-\frac{2}{\pi_{0k}^3}\frac{\partial \pi_{0k}}{\partial \alpha_{0a}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}}\frac{\partial \pi_{0k}}{\partial \alpha_{0c}} + \frac{1}{\pi_{0k}^2}\sum_{(ab)}^{2}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0c}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}}\right),$$

$$\frac{\partial^2 g_{0ab}}{\partial \pi_{\mathrm{T}k*}\partial \pi_{\mathrm{T}l}} = 0,$$

$$\frac{\partial^2 g_{0ab}}{\partial \pi_{\mathrm{T}k*}\partial \alpha_{0c}} = -\frac{2}{\pi_{0k*}^3}\frac{\partial \pi_{0k*}}{\partial \alpha_{0a}}\frac{\partial \pi_{0k*}}{\partial \alpha_{0b}}\frac{\partial \pi_{0k*}}{\partial \alpha_{0c}} + \frac{1}{\pi_{0k*}^2}\sum_{(ab)}^{2}\frac{\partial^2 \pi_{0k*}}{\partial \alpha_{0a}\partial \alpha_{0c}}\frac{\partial \pi_{0k*}}{\partial \alpha_{0b}},$$

$$\frac{\partial^2 g_{0ab}}{\partial \alpha_{0c}\partial \alpha_{0d*}} = \sum_{k=1}^{K}\pi_{\mathrm{T}k}\left\{\frac{6}{\pi_{0k}^4}\frac{\partial \pi_{0k}}{\partial \alpha_{0a}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}}\frac{\partial \pi_{0k}}{\partial \alpha_{0c}}\frac{\partial \pi_{0k}}{\partial \alpha_{0d*}}\right.$$

$$-\frac{2}{\pi_{0k}^3}\left(\sum_{(abcd*)}^{4}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0c}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}}\frac{\partial \pi_{0k}}{\partial \alpha_{0d*}} + \frac{\partial \pi_{0k}}{\partial \alpha_{0a}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0c}\partial \alpha_{0d*}}\right) \qquad \text{(S.14)}$$

$$+\frac{1}{\pi_{0k}^2}\left(\sum_{(ab)}^{2}\frac{\partial^3 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0c}\partial \alpha_{0d*}}\frac{\partial \pi_{0k}}{\partial \alpha_{0b}} + \frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0c}}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0b}\partial \alpha_{0d*}}\right.$$

$$\left.\left.+\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0a}\partial \alpha_{0d*}}\frac{\partial^2 \pi_{0k}}{\partial \alpha_{0b}\partial \alpha_{0c}}\right)\right\}$$

$$(a,b,c,d^* = 1,...,q; k^*, l = 1,...,K).$$

(d) t_R

Define $\hat{r}_{\mathrm{W}\alpha\alpha}$ as

$$t_R \equiv \frac{N^{1/2}(\hat{\alpha}_{\mathrm{W}} - \alpha_0)}{\left(\dfrac{\partial\hat{\alpha}_{\mathrm{W}}}{\partial\mathbf{p}'}\,\hat{\mathbf{\Omega}}_{\mathrm{T}}\,\dfrac{\partial\hat{\alpha}_{\mathrm{W}}}{\partial\mathbf{p}}\right)^{1/2}} \equiv \frac{N^{1/2}(\hat{\alpha}_{\mathrm{W}} - \alpha_0)}{\hat{r}_{\mathrm{W}\alpha\alpha}^{1/2}}, \tag{S.15}$$

where $\dfrac{\partial\hat{\alpha}_{\mathrm{W}}}{\partial\mathbf{p}} = \dfrac{\partial\alpha_{\mathrm{W}}(\mathbf{p})}{\partial\mathbf{p}}\Big|_{\mathbf{p}=\mathbf{p}}$.

Recalling $\dfrac{\partial\alpha_{\mathrm{W}}}{\partial\boldsymbol{\pi}_{\mathrm{T}}} = \dfrac{\partial\alpha_{\mathrm{W}}(\mathbf{p})}{\partial\mathbf{p}}\Big|_{\substack{\boldsymbol{\alpha}=\boldsymbol{\alpha}_{\mathrm{W}}\\ \mathbf{p}=\boldsymbol{\pi}_{\mathrm{T}}}} = \dfrac{\partial\alpha_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}} + N^{-1}\dfrac{\partial\alpha_{\Delta\mathrm{W}}}{\partial\boldsymbol{\pi}_{\mathrm{T}}} + O(N^{-2})$ (see (3.4)),

and using the definitions

$$r_{\mathrm{W}\alpha\alpha} \equiv \frac{\partial\alpha_{\mathrm{W}}}{\partial\boldsymbol{\pi}_{\mathrm{T}}'}\mathbf{\Omega}_{\mathrm{T}}\frac{\partial\alpha_{\mathrm{W}}}{\partial\boldsymbol{\pi}_{\mathrm{T}}}, \quad r_{0\alpha\alpha} \equiv \frac{\partial\alpha_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}'}\mathbf{\Omega}_{\mathrm{T}}\frac{\partial\alpha_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}} \tag{S.16}$$

and $r_{\Delta\mathrm{W}0} \equiv \dfrac{\partial\alpha_{\Delta\mathrm{W}}}{\partial\boldsymbol{\pi}_{\mathrm{T}}'}\mathbf{\Omega}_{\mathrm{T}}\dfrac{\partial\alpha_0}{\partial\boldsymbol{\pi}_{\mathrm{T}}}$,

we have

$$\hat{r}_{\mathrm{W}\alpha\alpha}^{-1/2} = r_{\mathrm{W}\alpha\alpha}^{-1/2} - \frac{1}{2}r_{\mathrm{W}\alpha\alpha}^{-3/2}\frac{\partial r_{\mathrm{W}ab}}{\partial\boldsymbol{\pi}_{\mathrm{T}}'}(\mathbf{p}-\boldsymbol{\pi}_{\mathrm{T}}) + \frac{3}{8}r_{\mathrm{W}\alpha\alpha}^{-5/2}\left(\frac{\partial r_{\mathrm{W}ab}}{\partial\boldsymbol{\pi}_{\mathrm{T}}'}\right)^{<2>}(\mathbf{p}-\boldsymbol{\pi}_{\mathrm{T}})^{<2>}$$

$$-\frac{1}{4}r_{\mathrm{W}\alpha\alpha}^{-3/2}\frac{\partial^2 r_{\mathrm{W}ab}}{(\partial\boldsymbol{\pi}_{\mathrm{T}}')^{<2>}}(\mathbf{p}-\boldsymbol{\pi}_{\mathrm{T}})^{<2>} + O_p(N^{-3/2}), \tag{S.17}$$

where

$$r_{\mathrm{W}\alpha\alpha} = r_{0\alpha\alpha} + 2N^{-1}r_{\Delta\mathrm{W}0} + O(N^{-2}),$$

$$r_{\mathrm{W}\alpha\alpha}^{-1/2} = r_{0\alpha\alpha}^{-1/2} - N^{-1}r_{0\alpha\alpha}^{-3/2}r_{\Delta\mathrm{W}0} + O(N^{-2}),$$

$$r_{\mathrm{W}\alpha\alpha}^{-j/2} = r_{0\alpha\alpha}^{-j/2} + O(N^{-1}) \ \ (j = 3,5), \tag{S.18}$$

$$\frac{\partial r_{\mathrm{W}ab}}{\partial\boldsymbol{\pi}_{\mathrm{T}}} = \frac{\partial r_{0ab}}{\partial\boldsymbol{\pi}_{\mathrm{T}}} + O(N^{-1}), \quad \frac{\partial^2 r_{\mathrm{W}ab}}{(\partial\boldsymbol{\pi}_{\mathrm{T}})^{<2>}} = \frac{\partial^2 r_{0ab}}{(\partial\boldsymbol{\pi}_{\mathrm{T}})^{<2>}} + O(N^{-1}).$$

From the above results, (S.17) becomes

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$$= r_{0\alpha\alpha}^{-1/2} - N^{-1} r_{0\alpha\alpha}^{-3/2} r_{\Delta W0} - \frac{N^{-1/2}}{2} r_{0\alpha\alpha}^{-3/2} \frac{\partial r_{0ab}}{\partial \boldsymbol{\pi}_{\mathrm{T}}}{}' \mathbf{m}^{(1)}$$

$$+ N^{-1} \left\{ \frac{3}{8} r_{0\alpha\alpha}^{-5/2} \left(\frac{\partial r_{0\alpha\alpha}}{\partial \boldsymbol{\pi}_{\mathrm{T}}}{}' \right)^{<2>} - \frac{1}{4} r_{0\alpha\alpha}^{-3/2} \frac{\partial^2 r_{0\alpha\alpha}}{(\partial \boldsymbol{\pi}_{\mathrm{T}}{}')^{<2>}} \right\} \mathbf{m}^{(1)<2>} + O_p(N^{-3/2}) \quad \text{(S.19)}$$

$$\equiv r_{0\alpha\alpha}^{-1/2} + N^{-1} \eta_{R0} + \sum_{j=1}^{2} N^{-j/2} \boldsymbol{\lambda}_R^{(j)}{}' \mathbf{m}^{(j)} + O_p(N^{-3/2}),$$

where $\eta_{R0} = -r_{0\alpha\alpha}^{-3/2} r_{\Delta W0},\ \boldsymbol{\lambda}_R^{(1)} = -\frac{1}{2} r_{0\alpha\alpha}^{-3/2} \dfrac{\partial r_{0\alpha\alpha}}{\partial \boldsymbol{\pi}_{\mathrm{T}}}$

and $\boldsymbol{\lambda}_R^{(2)} = \dfrac{3}{8} r_{0\alpha\alpha}^{-5/2} \left(\dfrac{\partial r_{0\alpha\alpha}}{\partial \boldsymbol{\pi}_{\mathrm{T}}} \right)^{<2>} - \dfrac{1}{4} r_{0\alpha\alpha}^{-3/2} \dfrac{\partial^2 r_{0\alpha\alpha}}{(\partial \boldsymbol{\pi}_{\mathrm{T}})^{<2>}}.$

In (S.19), using $\dfrac{\partial \boldsymbol{\Omega}_{\mathrm{T}}}{\partial \pi_{\mathrm{T}j}} = \mathbf{E}_{jj} - \boldsymbol{\pi}_{\mathrm{T}} \mathbf{e}_j{}' - \mathbf{e}_j \boldsymbol{\pi}_{\mathrm{T}}{}'$ where $\mathbf{e}_j$ is the vector

whose j-th element is 1 with other ones being zero, we have

$$\frac{\partial r_{0\alpha\alpha}}{\partial \pi_{\mathrm{T}j}} = 2 \frac{\partial^2 \alpha_0}{\partial \pi_{\mathrm{T}j} \partial \boldsymbol{\pi}_{\mathrm{T}}}{}' \boldsymbol{\Omega}_{\mathrm{T}} \frac{\partial \alpha_0}{\partial \boldsymbol{\pi}_{\mathrm{T}}} + \left(\frac{\partial \alpha_0}{\partial \pi_{\mathrm{T}j}} \right)^2 - 2 \frac{\partial \alpha_0}{\partial \pi_{\mathrm{T}j}} \boldsymbol{\pi}_{\mathrm{T}}{}' \frac{\partial \alpha_0}{\partial \boldsymbol{\pi}_{\mathrm{T}}},$$

$$\frac{\partial^2 r_{0\alpha\alpha}}{\partial \pi_{\mathrm{T}j} \partial \pi_{\mathrm{T}k}} = 2 \frac{\partial^3 \alpha_0}{\partial \pi_{\mathrm{T}j} \partial \pi_{\mathrm{T}k} \partial \boldsymbol{\pi}_{\mathrm{T}}}{}' \boldsymbol{\Omega}_{\mathrm{T}} \frac{\partial \alpha_0}{\partial \boldsymbol{\pi}_{\mathrm{T}}} + 2 \frac{\partial^2 \alpha_0}{\partial \pi_{\mathrm{T}j} \partial \boldsymbol{\pi}_{\mathrm{T}}}{}' \boldsymbol{\Omega}_{\mathrm{T}} \frac{\partial^2 \alpha_0}{\partial \pi_{\mathrm{T}k} \partial \boldsymbol{\pi}_{\mathrm{T}}} \quad \text{(S.20)}$$

$$+ \sum_{(jk)}^{2} 2 \left(\frac{\partial^2 \alpha_0}{\partial \pi_{\mathrm{T}j} \partial \pi_{\mathrm{T}k}} \frac{\partial \alpha_0}{\partial \pi_{\mathrm{T}k}} - \frac{\partial^2 \alpha_0}{\partial \pi_{\mathrm{T}j} \partial \boldsymbol{\pi}_{\mathrm{T}}}{}' \boldsymbol{\pi}_{\mathrm{T}} \frac{\partial \alpha_0}{\partial \pi_{\mathrm{T}k}} \right)$$

$$- 2 \frac{\partial \alpha_0}{\partial \pi_{\mathrm{T}j}} \frac{\partial \alpha_0}{\partial \pi_{\mathrm{T}k}} - 4 \frac{\partial^2 \alpha_0}{\partial \pi_{\mathrm{T}j} \partial \pi_{\mathrm{T}k}} \boldsymbol{\pi}_{\mathrm{T}}{}' \frac{\partial \alpha_0}{\partial \boldsymbol{\pi}_{\mathrm{T}}} \quad (j,k=1,...,q).$$

(e) $\eta_{\text{B-C}V0}$

The quantity η_{V0} is the generic expression for $\eta_{t0}(=\mathbf{i}_0^{(1)}{}'\boldsymbol{\eta}_0)$,
$\eta_{H0}(=\mathbf{d}_0^{(1\text{B})}{}'\boldsymbol{\eta}_0)$, $\eta_{G0}(=\mathbf{g}_0^{(1\text{B})}{}'\boldsymbol{\eta}_0)$ and

$$\eta_{R0}\left(=-r_{0\alpha\alpha}^{-3/2}r_{\Delta W0}=-r_{0\alpha\alpha}^{-3/2}\frac{\partial\alpha_{\Delta W}}{\partial\boldsymbol{\pi}_{\text{T}}{}'}\boldsymbol{\Omega}_{\text{T}}\frac{\partial\alpha_0}{\partial\boldsymbol{\pi}_{\text{T}}}\right).$$ The corresponding $\eta_{\text{B-C}V0}$ is

generically defined for $\eta_{\text{B-C}\,t0}(=\mathbf{i}_0^{(1)}{}'\boldsymbol{\eta}_{\text{B-C}\,0})$, $\eta_{\text{B-C}\,H0}(=\mathbf{d}_0^{(1\text{B})}{}'\boldsymbol{\eta}_{\text{B-C}\,0})$,

$$\eta_{\text{B-C}\,G0}(=\mathbf{g}_0^{(1\text{B})}{}'\boldsymbol{\eta}_{\text{B-C}\,0})\ \text{ and }\ \eta_{\text{B-C}\,R0}\left(=r_{0\alpha\alpha}^{-3/2}\frac{\partial\beta_{\text{ML1}}}{\partial\boldsymbol{\pi}_{\text{T}}{}'}\boldsymbol{\Omega}_{\text{T}}\frac{\partial\alpha_0}{\partial\boldsymbol{\pi}_{\text{T}}}\right).$$

(f) A simplification of Theorem 6.

$\kappa_2(w_{\text{B-P}}^*)$ can be simplified as follows. Using (8.20), (8.19) becomes

$$\frac{\partial\boldsymbol{\alpha}_{\text{B-P}}}{\partial\boldsymbol{\pi}_{\text{T}}{}'}=-\boldsymbol{\Lambda}^{-1}\frac{\partial^2 l_{\text{ML}}}{\partial\boldsymbol{\alpha}_0\partial\boldsymbol{\pi}_{\text{T}}{}'}-N^{-1}\frac{\partial\boldsymbol{\beta}_{\text{ML1}}}{\partial\boldsymbol{\pi}_{\text{T}}{}'}+O(N^{-2})$$

$$\equiv\frac{\partial\boldsymbol{\alpha}_0}{\partial\boldsymbol{\pi}_{\text{T}}{}'}+N^{-1}\frac{\partial\boldsymbol{\alpha}_{\Delta\text{B-P}}}{\partial\boldsymbol{\pi}_{\text{T}}{}'}+O(N^{-2}),$$

which gives

$$\kappa_2(w_{\text{B-P}}^*)=\beta_{\text{ML2}}+N^{-1}\left(\beta_{\text{ML}\Delta2}+2\frac{\partial\alpha_0}{\partial\boldsymbol{\pi}_{\text{T}}{}'}\boldsymbol{\Omega}_{\text{T}}\frac{\partial\alpha_{\Delta\text{B-P}}}{\partial\boldsymbol{\pi}_{\text{T}}}\right)+O(N^{-2})$$

$$\left(\beta_{\text{B-P2}}=\beta_{\text{ML2}},\ \beta_{\text{B-P}\Delta2}=\beta_{\text{B-CML}\Delta2}=\beta_{\text{ML}\Delta2}-2\frac{\partial\alpha_0}{\partial\boldsymbol{\pi}_{\text{T}}{}'}\boldsymbol{\Omega}_{\text{T}}\frac{\partial\beta_{\text{ML1}}}{\partial\boldsymbol{\pi}_{\text{T}}}\right).$$

That is, we find that the asymptotic cumulants of $\hat{\alpha}_{\text{B-P}}$ up to the fourth order
and the higher-order asymptotic variance are identical to those of $\hat{\alpha}_{\text{B-CML}}$ or
the bias-corrected ML estimator, respectively.

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