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Supplement to the paper "Asymptotic expansions for the ability estimator in item response theory"

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This note is to supplement Ogasawara (2012).

1. **Expansions of** $\hat{h}^{1/2}$, $\hat{g}^{1/2}$ **and** $\hat{h}\hat{g}^{-1/2}$

For the expansion of $\hat{h}^{1/2}$, recalling

$$\hat{\theta}-\theta_0=\lambda^{(1)}l_0^{(1)}+\boldsymbol{\lambda}^{(2)}{}'\mathbf{l}_0^{(2)}+O_p(n^{-3/2})$$

$$=-\lambda^{-1}\frac{\partial\overline{l}}{\partial\theta_0}+\lambda^{-2}m\frac{\partial\overline{l}}{\partial\theta_0}-\frac{\lambda^{-1}}{2}\mathrm{E}_{\mathrm{T}}(j_0^{(3)})\left(\lambda^{-1}\frac{\partial\overline{l}}{\partial\theta_0}\right)^2+O_p(n^{-3/2}),$$

we have

$$\hat{h}^{1/2}=h_0^{1/2}+\frac{h_0^{-1/2}}{2}\frac{\partial h_0}{\partial\theta_0}(\hat{\theta}-\theta_0)+\left\{\frac{h_0^{-1/2}}{4}\frac{\partial^2 h_0}{\partial\theta_0^2}-\frac{h_0^{-3/2}}{8}\left(\frac{\partial h_0}{\partial\theta_0}\right)^2\right\}(\hat{\theta}-\theta_0)^2$$

$$+O_p(n^{-3/2})$$

$$=(-\lambda)^{1/2}-\frac{(-\lambda)^{-1/2}}{2}m-\frac{(-\lambda)^{-3/2}}{8}m^2+\frac{1}{2}\left\{(-\lambda)^{-1/2}+\frac{(-\lambda)^{-3/2}}{2}m\right\}\frac{\partial h_0}{\partial\theta_0}$$

$$\times(\lambda^{(1)}l_0^{(1)}+\boldsymbol{\lambda}^{(2)}{}'\mathbf{l}_0^{(2)})$$

$$+\left[\frac{(-\lambda)^{-1/2}}{4}\mathrm{E}_{\mathrm{T}}\left(\frac{\partial^2 h_0}{\partial\theta_0^2}\right)-\frac{(-\lambda)^{-3/2}}{8}\left\{\mathrm{E}_{\mathrm{T}}\left(\frac{\partial h_0}{\partial\theta_0}\right)\right\}^2\right](\lambda^{(1)}l_0^{(1)})^2$$

$$+O_p(n^{-3/2})$$

$$= (-\lambda)^{1/2} + \left\{ -\frac{(-\lambda)^{-1/2}}{2}, -\frac{(-\lambda)^{-3/2}}{2} \mathrm{E_T}(j_0^{(3)}) \right\} (m, l_0^{(1)})'$$

$$+ \left[-\frac{(-\lambda)^{-3/2}}{8}, -\frac{3}{4}(-\lambda)^{-5/2} \mathrm{E_T}(j_0^{(3)}), -\frac{3}{8}(-\lambda)^{-7/2} \{\mathrm{E_T}(j_0^{(3)})\}^2 \right.$$

$$\left. -\frac{(-\lambda)^{-5/2}}{4} \mathrm{E_T}(j_0^{(4)}), -\frac{(-\lambda)^{-3/2}}{2} \right] [m^2, ml_0^{(1)}, (l_0^{(1)})^2, \{j_0^{(3)} - \mathrm{E_T}(j_0^{(3)})\} l_0^{(1)}]'$$

$$+ O_p(n^{-3/2})$$

$$\equiv (-\lambda)^{1/2} + \sum_{i=1}^{2} \mathbf{h}^{(i)}{}' \mathbf{m}_0^{(i)} + O_p(n^{-3/2}) \,.$$

For $\hat{g}^{1/2}$,

$$\hat{g}^{1/2} = g_0^{1/2} + \frac{g_0^{-1/2}}{2} \frac{\partial g_0}{\partial \theta_0} (\hat{\theta} - \theta_0) + \left\{ \frac{g_0^{-1/2}}{4} \frac{\partial^2 g_0}{\partial \theta_0^2} - \frac{g_0^{-3/2}}{8} \left(\frac{\partial g_0}{\partial \theta_0} \right)^2 \right\} (\hat{\theta} - \theta_0)^2$$

$$+ O_p(n^{-3/2}).$$

Recalling $\gamma = \mathrm{E_T}(g_0)$ and $g_0 = \gamma + m_g$ with $m_g = O_p(n^{-1/2})$, the above result becomes

$$= \gamma^{1/2} + \frac{\gamma^{-1/2}}{2} m_g - \frac{\gamma^{-3/2}}{8} m_g^2 + \frac{1}{2} \left\{ \gamma^{-1/2} - \frac{\gamma^{-3/2}}{2} m_g \right\} \frac{\partial g_0}{\partial \theta_0} (\lambda^{(1)} l_0^{(1)} + \boldsymbol{\lambda}^{(2)}{}' \mathbf{l}_0^{(2)})$$

$$+ \left[\frac{\gamma^{-1/2}}{4} \mathrm{E_T}\left(\frac{\partial^2 g_0}{\partial \theta_0^2} \right) - \frac{\gamma^{-3/2}}{8} \left\{ \mathrm{E_T}\left(\frac{\partial g_0}{\partial \theta_0} \right) \right\}^2 \right] (\lambda^{(1)} l_0^{(1)})^2 + O_p(n^{-3/2})$$

$$= \gamma^{1/2} + \left\{ \frac{\gamma^{-1/2}}{2}, -\frac{\gamma^{-1/2}}{2} \lambda^{-1} \mathrm{E_T}\left(\frac{\partial g_0}{\partial \theta_0} \right) \right\} (m_g, l_0^{(1)})'$$

$$+\left[-\frac{\gamma^{-3/2}}{8},\frac{\gamma^{-3/2}}{4}\lambda^{-1}\mathrm{E_T}\left(\frac{\partial g_0}{\partial\theta_0}\right),\frac{\gamma^{-1/2}}{2}\lambda^{-2}\mathrm{E_T}\left(\frac{\partial g_0}{\partial\theta_0}\right),\right.$$

$$-\frac{\gamma^{-1/2}}{4}\lambda^{-3}\mathrm{E_T}\left(\frac{\partial g_0}{\partial\theta_0}\right)\mathrm{E_T}(j_0^{(3)})$$

$$\left.+\frac{\gamma^{-1/2}}{4}\lambda^{-2}\mathrm{E_T}\left(\frac{\partial^2 g_0}{\partial\theta_0^2}\right)-\frac{\gamma^{-3/2}}{8}\lambda^{-2}\left\{\mathrm{E_T}\left(\frac{\partial g_0}{\partial\theta_0}\right)\right\}^2,-\frac{\gamma^{-1/2}}{2}\lambda^{-1}\right]$$

$$\times\left[m_g^2,m_g l_0^{(1)},ml_0^{(1)},(l_0^{(1)})^2,\left\{\frac{\partial g_0}{\partial\theta_0}-\mathrm{E_T}\left(\frac{\partial g_0}{\partial\theta_0}\right)\right\}l_0^{(1)}\right]'+O_p(n^{-3/2})$$

$$\equiv\gamma^{1/2}+\sum_{i=1}^{2}\mathbf{g}^{(i)}\,'\mathbf{m}_{g0}^{(i)}+O_p(n^{-3/2}).$$

For $\hat{h}\hat{g}^{-1/2}$,

$$\hat{h}\hat{g}^{-1/2}=h_0 g_0^{-1/2}+\left(\frac{\partial h_0}{\partial\theta_0}g_0^{-1/2}-\frac{h_0}{2}g_0^{-3/2}\frac{\partial g_0}{\partial\theta_0}\right)(\hat{\theta}-\theta_0)+\frac{1}{2}\left\{-g_0^{-3/2}\frac{\partial h_0}{\partial\theta_0}\frac{\partial g_0}{\partial\theta_0}\right.$$

$$\left.+g_0^{-1/2}\frac{\partial^2 h_0}{\partial\theta_0^2}-\frac{h_0}{2}g_0^{-3/2}\frac{\partial^2 g_0}{\partial\theta_0^2}+\frac{3}{4}h_0 g_0^{-5/2}\left(\frac{\partial g_0}{\partial\theta_0}\right)^2\right\}(\hat{\theta}-\theta_0)^2$$

$$+O_p(n^{-3/2})$$

$$=-(\lambda+m)\left(\gamma^{-1/2}-\frac{\gamma^{-3/2}}{2}m_g+\frac{3}{8}\gamma^{-5/2}m_g^2\right)+\left\{-j_0^{(3)}\left(\gamma^{-1/2}-\frac{\gamma^{-3/2}}{2}m_g\right)\right.$$

$$\left.+\frac{1}{2}(\lambda+m)\left(\gamma^{-3/2}-\frac{3}{2}\gamma^{-5/2}m_g\right)\frac{\partial g_0}{\partial\theta_0}\right\}(\lambda^{(1)}l_0^{(1)}+\boldsymbol{\lambda}^{(2)}\,'\mathbf{l}_0^{(2)})$$

$$+\frac{1}{2}\left[\gamma^{-3/2}\mathrm{E_T}(j_0^{(3)})\mathrm{E_T}\left(\frac{\partial g_0}{\partial\theta_0}\right)-\gamma^{-1/2}\mathrm{E_T}(j_0^{(4)})\right.$$

$$\left.+\frac{\lambda}{2}\gamma^{-3/2}\mathrm{E_T}\left(\frac{\partial^2 g_0}{\partial\theta_0^2}\right)-\frac{3}{4}\lambda\gamma^{-5/2}\left\{\mathrm{E_T}\left(\frac{\partial g_0}{\partial\theta_0}\right)\right\}^2\right](\lambda^{(1)}l_0^{(1)})^2$$

$$+O_p(n^{-3/2})$$

$$
\begin{aligned}
&= -\lambda\gamma^{-1/2} + \left\{ -\gamma^{-1/2},\ \frac{\gamma^{-3/2}}{2}\lambda,\ \mathrm{E_T}(j_0^{(3)})\gamma^{-1/2}\lambda^{-1} - \frac{\gamma^{-3/2}}{2}\mathrm{E_T}\left(\frac{\partial g_0}{\partial \theta_0}\right) \right\} \\
&\qquad\qquad \times (m, m_g, l_0^{(1)})' \\
&+ \left[-\frac{3}{8}\gamma^{-5/2}\lambda,\ \frac{\gamma^{-3/2}}{2}, -\mathrm{E_T}(j_0^{(3)})\gamma^{-1/2}\lambda^{-2}, -\frac{\gamma^{-3/2}}{2}\lambda^{-1}\mathrm{E_T}(j_0^{(3)}) \right. \\
&\qquad\quad + \frac{3}{4}\gamma^{-5/2}\mathrm{E_T}\left(\frac{\partial g_0}{\partial \theta_0}\right),\ \gamma^{-1/2}\lambda^{-1}, \\
&\quad -\frac{\gamma^{-3/2}}{2}, \frac{\gamma^{-1/2}}{2}\lambda^{-3}\{\mathrm{E_T}(j_0^{(3)})\}^2 + \frac{\gamma^{-3/2}}{4}\lambda^{-2}\mathrm{E_T}(j_0^{(3)})\mathrm{E_T}\left(\frac{\partial g_0}{\partial \theta_0}\right) \\
&\quad \left. -\frac{\gamma^{-1/2}}{2}\lambda^{-2}\{\mathrm{E_T}(j_0^{(4)})\} + \frac{\gamma^{-3/2}}{4}\lambda^{-1}\mathrm{E_T}\left(\frac{\partial^2 g_0}{\partial \theta_0^2}\right) - \frac{3}{8}\gamma^{-5/2}\lambda^{-1}\left\{\mathrm{E_T}\left(\frac{\partial g_0}{\partial \theta_0}\right)\right\}^2 \right] \\
&\times \left[m_g^2, mm_g, ml_0^{(1)}, m_g l_0^{(1)}, \mathrm{E_T}\{j_0^{(3)} - \mathrm{E_T}(j_0^{(3)})\}l_0^{(1)}, \left\{\frac{\partial g_0}{\partial \theta_0} - \mathrm{E_T}\left(\frac{\partial g_0}{\partial \theta_0}\right)\right\} l_0^{(1)}, (l_0^{(1)})^2 \right]' \\
&+ O_p(n^{-3/2}) \\
&\equiv -\lambda\gamma^{-1/2} + \sum_{i=1}^{2} \mathbf{r}^{(i)}{}'\mathbf{m}_{r0}^{(i)} + O_p(n^{-3/2}).
\end{aligned}
$$

2. Supplementary numerical results

Table A gives the values of bias, skewness and kurtosis for *t* under model misspecification. Figures 1 and 2 plot the empirical and asymptotic distributions of *t* for $\hat{\theta}$ when n=50 and θ=0, 1, 2 and t_h for $\exp(\hat{\theta})$ when n=50, 100, 300, 800 and θ=2, respectively. In the plot of θ=2 in Figure 1, some discrepancy between the normal approximation (dashed lines) and the empirical distribution (histogram) is observed. The discrepancy is more pronounced in Figure 2 when *n* is relatively small.

Table B shows the errors of the distribution functions approximated by the usual normal (N*), the single- (E1) and two-term (E2) Edgeworth expansions (see Theorems 1 and 2), and Hall’s (1992b) method of variable transformation, which is asymptotically as accurate as E1, under selected conditions with or without model misspecification. The true or empirical distributions were given from the same simulations shown earlier. The values of t and t_r are taken at $-3.8(0.2)4.0$. It is found that E1, E2 and Hall have reduced the errors of N* and that E2 on average has reduced the errors of E1 and Hall.

Reference

Ogasawara, H. (2012). Asymptotic expansions for the ability estimator in item response theory. *Computational Statistics*, *27*, 661-683.

Table A. Simulated and asymptotic cumulants when the IRT model is false

n=300		Slight misspecification				Gross misspecification			
		θ = -1		θ =2		θ = -1		θ =2	
t									
		Sim.	Th.	Sim	Th.	Sim.	Th.	Sim	Th.
α_1	$\hat{\theta}$	.58	.54	.02	.08	.53	.49	.03	.04
	P.C.	-.27	-.28	.94	.96	-.19	-.22	.83	.81
	logistic	-.02	-.05	.95	.98	.02	-.01	.84	.83
	exp($\hat{\theta}$)	-.70	-.73	-1.17	-1.10	-.56	-.60	-1.01	-.98
α_3	$\hat{\theta}$	3.11	3.23	-1.71	-1.66	2.84	2.83	-1.47	-1.38
	P.C.	-1.73	-1.39	4.13	3.57	-.63	-.56	2.86	2.59
	logistic	-.21	-.07	4.06	3.69	.44	.41	2.83	2.68
	exp($\hat{\theta}$)	-4.15	-3.90	-8.92	-8.67	-2.43	-2.40	-6.91	-6.72
α_4	$\hat{\theta}$	-3	-2	-9	-10	7	5	-11	-6
	P.C.	25	19	55	43	12	10	28	27
	logistic	9	8	41	34	6	6	19	21
	exp($\hat{\theta}$)	35	30	109	99	13	13	64	70

Note. *n*=the number of items, Sim.=simulated value, Th.=theoretical or asymptotic value, P.C.=the estimator of the proportion-correct true score, logistic=1/{1+exp($-\hat{\theta}$)}.

Table B. $10^5 \times$ root mean square errors of the asymptotic distribution functions for the pivots

		True model								Gross misspecification			
		n=50, θ=0				n=300, θ=2				n=300, θ=2			
		N*	E1	E2	Hall	N*	E1	E2	Hall	N*	E1	E2	Hall
t	$\hat{\theta}$	477	88	89	103	837	167	104	156	316	80	87	80
	P.C.	250	222	97	221	820	179	103	166	760	172	89	171
	logistic	553	332	111	323	826	149	104	142	768	143	88	147
	exp($\hat{\theta}$)	1901	366	145	297	1241	201	127	239	1143	138	105	199
t_r	$\hat{\theta}$	530	384	161	385	259	109	103	109	316	110	92	107
	P.C.	610	601	169	600	830	214	107	202	869	204	89	203
	logistic	781	703	183	697	836	182	106	175	877	174	85	177
	exp($\hat{\theta}$)	2024	611	214	513	1230	182	114	221	978	113	95	166

Note. n=the number of items, N*=normal approximation, E1=single-term Edgeworth expansion, E2=two-term Edgeworth expansion, Hall=Hall's method involving variable transformation, P.C.=the estimator of the proportion-correct true score, logistic=1/{1+exp($-\hat{\theta}$)}.

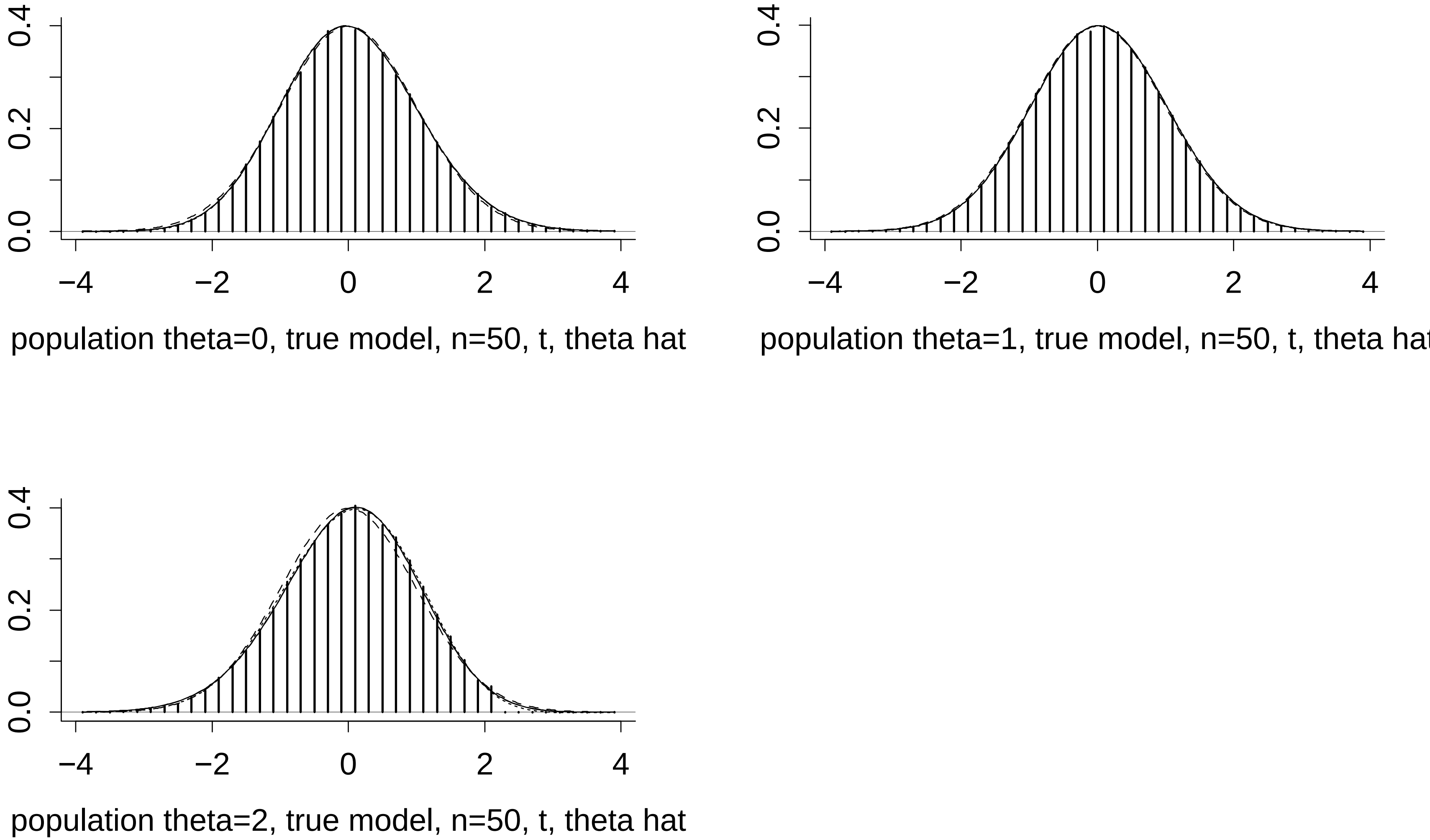

Figure 1. Theoretical (curved lines) and simulated (histograms) distributions of the pivot t (dashed lines=the normal distribution, solid lines=the single−term Edgeworth expansion, dotted lines=the two−term Edgeworth expansion, long dashed lines=Hall's variable transformation).

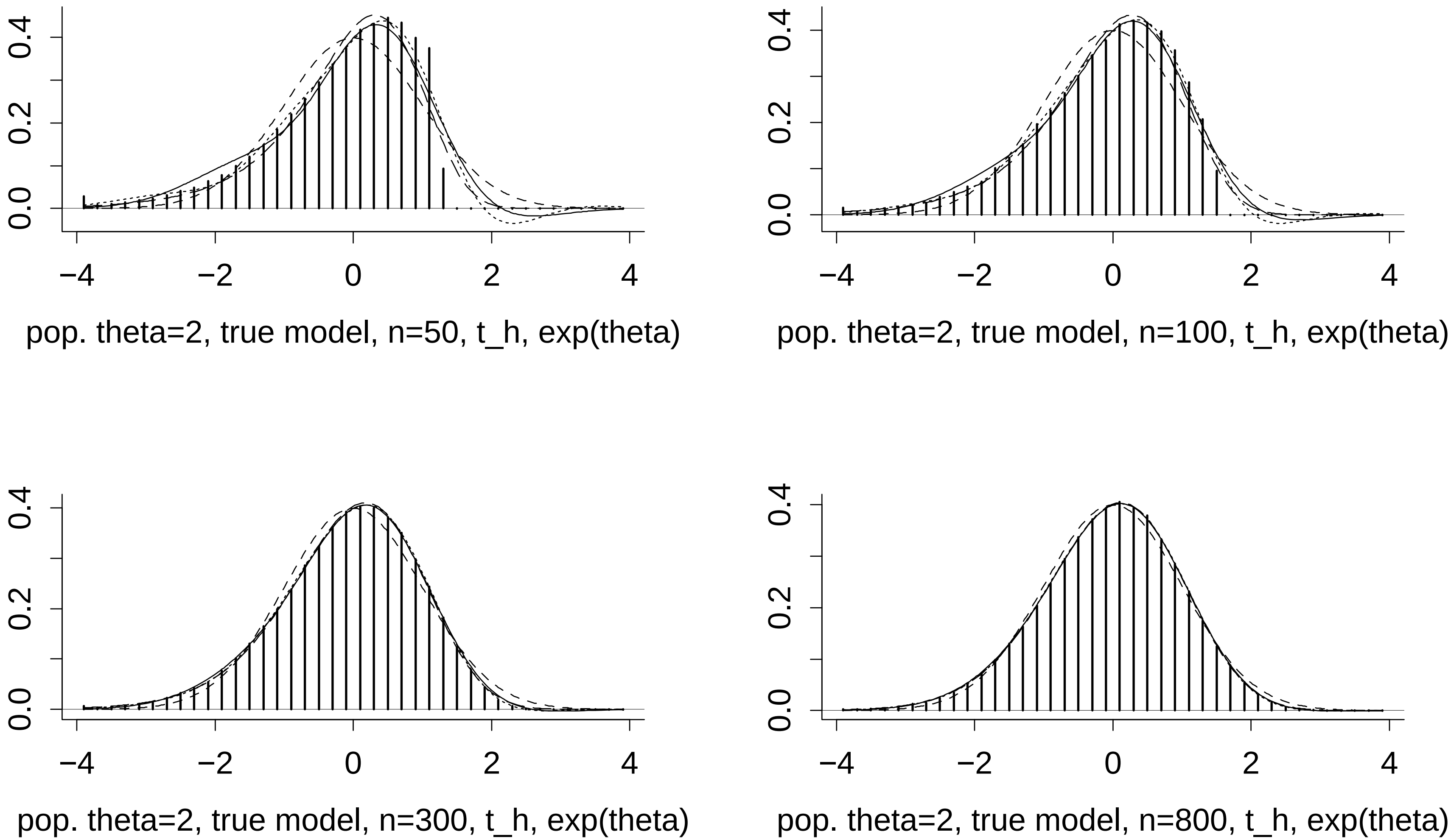

Figure 2. Theoretical (curved lines) and simulated (histograms) distributions of the pivot t_h (dashed lines=the normal distribution, solid lines=the single−term Edgeworth expansion, dotted lines=the two−term Edgeworth expansion, long dashed lines=Hall's variable transformation).