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Supplement to the paper "An asymptotic equivalence of the cross-data and predictive estimators"

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This article supplements Ogasawara (2019).

S1. Bernoulli distribution
S1.1 Bernoulli distribution under canonical parametrization

$$\Pr(x\,|\,\theta) = \left\{\frac{1}{1+\exp(-\theta)}\right\}^{x}\left\{\frac{\exp(-\theta)}{1+\exp(-\theta)}\right\}^{1-x} \quad (x=0,1),$$

$$= \frac{\exp\{-(1-x)\theta\}}{1+\exp(-\theta)}, \quad \left(\hat{\theta}_{\mathrm{ML}(-i)} = \ln\frac{\bar{x}_{(-i)}}{1-\bar{x}_{(-i)}}\right),$$

$$D_{(-)} = \sum_{i=1}^{n}\left[(1-x_i)\omega\,\hat{\theta}_{\mathrm{ML}(-i)} + \ln\{1+\exp(-\omega\,\hat{\theta}_{\mathrm{ML}(-i)})\}\right],$$

$$\frac{\partial D_{(-)}}{\partial\omega} = \sum_{i=1}^{n}\left\{(1-x_i)\,\hat{\theta}_{\mathrm{ML}(-i)} - \frac{\exp(-\omega\,\hat{\theta}_{\mathrm{ML}(-i)})}{1+\exp(-\omega\,\hat{\theta}_{\mathrm{ML}(-i)})}\hat{\theta}_{\mathrm{ML}(-i)}\right\},$$

$$= \sum_{i=1}^{n}\left\{1-x_i - \frac{\{\bar{x}_{(-i)}/(1-\bar{x}_{(-i)})\}^{-\omega}}{1+\{\bar{x}_{(-i)}/(1-\bar{x}_{(-i)})\}^{-\omega}}\right\}\ln\frac{\bar{x}_{(-i)}}{1-\bar{x}_{(-i)}} \qquad \text{(S1.1)}$$

$$= \sum_{i=1}^{n}\left\{\frac{1}{1+\{\bar{x}_{(-i)}/(1-\bar{x}_{(-i)})\}^{-\omega}} - x_i\right\}\ln\frac{\bar{x}_{(-i)}}{1-\bar{x}_{(-i)}},$$

where

$$\bar{x}_{(-i)} = \bar{x} - (n-1)^{-1}(x_i - \bar{x}) \ (i=1,\ldots,n)\,. \qquad \text{(S1.2)}$$

To solve $\partial D_{(-)}/\partial\omega = 0$, we use

$$\frac{\partial^2 D_{(-)}}{\partial \omega^2} = \sum_{i=1}^{n} \frac{\exp(-\omega \hat{\theta}_{\mathrm{ML}(-i)})}{\{1+\exp(-\omega \hat{\theta}_{\mathrm{ML}(-i)})\}^2} \hat{\theta}^2_{\mathrm{ML}(-i)}$$

$$= \sum_{i=1}^{n} \frac{\{\bar{x}_{(-i)}/(1-\bar{x}_{(-i)})\}^{-\omega}}{[1+\{\bar{x}_{(-i)}/(1-\bar{x}_{(-i)})\}^{-\omega}]^2} \left(\ln \frac{\bar{x}_{(-i)}}{1-\bar{x}_{(-i)}} \right)^2 . \tag{S1.3}$$

S1.2 Bernoulli distribution using the expectation parameter

$$\Pr(x \mid \pi) = \pi^x (1-\pi)^{1-x} \quad (x = 0,1), \quad (\hat{\pi}_{\mathrm{ML}(-i)} = \bar{x}_{(-i)}),$$

$$D_{(-)} = \sum_{i=1}^{n} \left\{ -x_i \ln(\omega \hat{\pi}_{\mathrm{ML}(-i)}) - (1-x_i) \ln(1-\omega \hat{\pi}_{\mathrm{ML}(-i)}) \right\}$$

$$= \sum_{i=1}^{n} \left\{ -x_i \ln(\omega \bar{x}_{(-i)}) - (1-x_i) \ln(1-\omega \bar{x}_{(-i)}) \right\},$$

$$\frac{\partial D_{(-)}}{\partial \omega} = \sum_{i=1}^{n} \left\{ -\frac{x_i}{\omega} + \frac{(1-x_i)\bar{x}_{(-i)}}{1-\omega \bar{x}_{(-i)}} \right\} = -\frac{n\bar{x}}{\omega} + \sum_{i=1}^{n} \frac{(1-x_i)\bar{x}_{(-i)}}{1-\omega \bar{x}_{(-i)}}, \tag{S1.4}$$

$$\frac{\partial^2 D_{(-)}}{\partial \omega^2} = \frac{n\bar{x}}{\omega^2} + \sum_{i=1}^{n} \frac{(1-x_i)\bar{x}^2_{(-i)}}{(1-\omega \bar{x}_{(-i)})^2}.$$

S2. Gamma distribution when the shape parameter α and the scale parameter β are unknown

$$f(x \mid \alpha, \beta) = \frac{x^{\alpha-1} \exp(-x/\beta)}{\beta^{\alpha} \Gamma(\alpha)} \quad (x > 0), \ \boldsymbol{\omega} = (\omega_\beta, \omega_\alpha)',$$

$$D_{(-)} = \sum_{i=1}^{n} \left\{ -(\omega_\alpha \hat{\alpha}_{\mathrm{ML}(-i)} - 1) \ln x_i + \omega_\alpha \hat{\alpha}_{\mathrm{ML}(-i)} \ln(\omega_\beta \hat{\beta}_{\mathrm{ML}(-i)}) + \frac{x_i}{\omega_\beta \hat{\beta}_{\mathrm{ML}(-i)}} + \ln \Gamma(\omega_\alpha \hat{\alpha}_{\mathrm{ML}(-i)}) \right\}, \tag{S2.1}$$

$$\frac{\partial D_{(-)}}{\partial \omega_\beta} = \sum_{i=1}^{n}\left(\frac{\omega_\alpha \hat{\alpha}_{\mathrm{ML}(-i)}}{\omega_\beta} - \frac{x_i}{\omega_\beta^2 \hat{\beta}_{\mathrm{ML}(-i)}}\right),\ (\hat{\beta}_{\mathrm{ML}(-i)} = \bar{x}_{(-i)} / \hat{\alpha}_{\mathrm{ML}(-i)}),$$

$$= \sum_{i=1}^{n}\left(\frac{\omega_\alpha \hat{\alpha}_{\mathrm{ML}(-i)}}{\omega_\beta} - \frac{\hat{\alpha}_{\mathrm{ML}(-i)} x_i}{\omega_\beta^2 \bar{x}_{(-i)}}\right),$$

$$\frac{\partial D_{(-)}}{\partial \omega_\alpha} = \sum_{i=1}^{n}\{-\hat{\alpha}_{\mathrm{ML}(-i)} \ln x_i + \hat{\alpha}_{\mathrm{ML}(-i)} \ln(\omega_\beta \hat{\beta}_{\mathrm{ML}(-i)}) + \psi(\omega_\alpha \hat{\alpha}_{\mathrm{ML}(-i)})\hat{\alpha}_{\mathrm{ML}(-i)}\},$$

$$(\psi(\alpha) = \partial \ln \Gamma(\alpha) / \partial \alpha).$$

From $\partial D_{(-)} / \partial \omega_\beta = 0$, we obtain

$$\hat{\omega}_\beta = \frac{\sum_{i=1}^{n} \hat{\alpha}_{\mathrm{ML}(-i)} x_i / \bar{x}_{(-i)}}{\hat{\omega}_\alpha \sum_{i=1}^{n} \hat{\alpha}_{\mathrm{ML}(-i)}}. \tag{S2.2}$$

From (S2.2) with $\hat{\beta}_{\mathrm{ML}(-i)} = \bar{x}_{(-i)} / \hat{\alpha}_{\mathrm{ML}(-i)}$, we have

$$\frac{\partial D_{(-)}}{\partial \omega_\alpha} = \sum_{i=1}^{n}\left\{-\hat{\alpha}_{\mathrm{ML}(-i)} \ln x_i + \hat{\alpha}_{\mathrm{ML}(-i)} \ln \frac{(\bar{x}_{(-i)} / \hat{\alpha}_{\mathrm{ML}(-i)}) \sum_{j=1}^{n} \hat{\alpha}_{\mathrm{ML}(-j)} x_j / \bar{x}_{(-j)}}{\omega_\alpha \sum_{j=1}^{n} \hat{\alpha}_{\mathrm{ML}(-j)}}\right.$$

$$\left. + \psi(\omega_\alpha \hat{\alpha}_{\mathrm{ML}(-i)})\hat{\alpha}_{\mathrm{ML}(-i)} \right\},$$

$$\frac{\partial^2 D_{(-)}}{\partial \omega_\alpha^2} = \sum_{i=1}^{n}\left\{-\frac{\hat{\alpha}_{\mathrm{ML}(-i)}}{\omega_\alpha} + \psi'(\omega_\alpha \hat{\alpha}_{\mathrm{ML}(-i)}^2)\hat{\alpha}_{\mathrm{ML}(-i)}^2\right\}, \tag{S2.3}$$

$$(\psi'(\alpha) = \partial \psi(\alpha) / \partial \alpha).$$

S3. Multivariate multiple regression

S3.1 Multivariate multiple regression with normal errors when regression coefficients and error variances/covariances are unknown

It is assumed that

$$\mathbf{y}_i \sim \mathrm{N}(\mathbf{B}\mathbf{x}_i, \mathbf{\Sigma})\ (i = 1,\ldots,n), \tag{S3.1}$$

where $\mathbf{y}_i$ is the $q^* \times 1$ response vector; **B** is the $q^* \times p^*$ matrix of regression coefficients; $\mathbf{x}_i$ is the $p^* \times 1$ vector of fixed covariates; and $\mathbf{\Sigma}$ is the $q^* \times q^*$ covariance matrix of $\mathbf{y}_i$.

For the cross-data estimators of $\mathbf{B}$ and $\boldsymbol{\Sigma}$, we have

$$\boldsymbol{\omega} = \{\mathrm{vec}'(\boldsymbol{\Omega}_{\mathrm{B}}), \mathrm{vec}'(\boldsymbol{\Omega}_{\Sigma})\}',$$

$$L_{(-)} \equiv \prod_{i=1}^{n} \frac{1}{(2\pi)^{q^*/2} \,|\boldsymbol{\Omega}_{\Sigma} \odot \hat{\boldsymbol{\Sigma}}_{\mathrm{ML}(-i)}|^{1/2}} \exp\{-(\mathbf{y}_i - \boldsymbol{\Omega}_{\mathrm{B}} \odot \hat{\mathbf{B}}_{\mathrm{ML}(-i)}\mathbf{x}_i)' \times (\boldsymbol{\Omega}_{\Sigma} \odot \hat{\boldsymbol{\Sigma}}_{\mathrm{ML}(-i)})^{-1}(\mathbf{y}_i - \boldsymbol{\Omega}_{\mathrm{B}} \odot \hat{\mathbf{B}}_{\mathrm{ML}(-i)}\mathbf{x}_i)/2\},$$

$$D_{(-)} = -\ln L_{(-)} = \frac{nq^*}{2}\ln(2\pi) + \frac{1}{2}\sum_{i=1}^{n}\ln|\boldsymbol{\Omega}_{\Sigma} \odot \hat{\boldsymbol{\Sigma}}_{\mathrm{ML}(-i)}| + \frac{1}{2}\sum_{i=1}^{n}(\mathbf{y}_i - \boldsymbol{\Omega}_{\mathrm{B}} \odot \hat{\mathbf{B}}_{\mathrm{ML}(-i)}\mathbf{x}_i)'(\boldsymbol{\Omega}_{\Sigma} \odot \hat{\boldsymbol{\Sigma}}_{\mathrm{ML}(-i)})^{-1} \times (\mathbf{y}_i - \boldsymbol{\Omega}_{\mathrm{B}} \odot \hat{\mathbf{B}}_{\mathrm{ML}(-i)}\mathbf{x}_i),$$

$$\frac{\partial D_{(-)}}{\partial \omega_{\mathrm{B}ab}} = -\sum_{i=1}^{n}\{(\boldsymbol{\Omega}_{\Sigma} \odot \hat{\boldsymbol{\Sigma}}_{\mathrm{ML}(-i)})^{-1}(\mathbf{y}_i - \boldsymbol{\Omega}_{\mathrm{B}} \odot \hat{\mathbf{B}}_{\mathrm{ML}(-i)}\mathbf{x}_i)\mathbf{x}_i'\}_{ab} \times (\hat{\mathbf{B}}_{\mathrm{ML}(-i)})_{ab},$$

$$(a = 1,\ldots,q^*;\ b = 1,\ldots,p^*),$$

$$\hat{\mathbf{B}}_{\mathrm{ML}(-i)} = (\mathbf{X}_{(-i)}'\mathbf{X}_{(-i)})^{-1}\mathbf{X}_{(-i)}'\mathbf{Y}_{(-i)}, \tag{S3.2}$$

where $\mathbf{X}_{(-i)} = (\mathbf{x}_1,\ldots,\mathbf{x}_n)'$ with $\mathbf{x}_i$ being deleted and $\mathbf{Y}_{(-i)}$ is similarly defined,

$$\hat{\boldsymbol{\Sigma}}_{\mathrm{ML}(-i)} = (n-1)^{-1}(\mathbf{Y}_{(-i)}' - \hat{\mathbf{B}}_{\mathrm{ML}(-i)}\mathbf{X}_{(-i)}')(\mathbf{Y}_{(-i)} - \mathbf{X}_{(-i)}\hat{\mathbf{B}}_{\mathrm{ML}(-i)}'),$$

$$\begin{aligned}\frac{\partial D_{(-)}}{\partial \omega_{\Sigma ef}} &= \frac{1}{2}\sum_{i=1}^{n}(2-\delta_{ef})\{(\boldsymbol{\Omega}_{\Sigma} \odot \hat{\boldsymbol{\Sigma}}_{\mathrm{ML}(-i)})^{-1}\}_{ef}\,\hat{\sigma}_{\mathrm{ML}(-i)ef} \\ &- \frac{1}{2}\sum_{i=1}^{n}(2-\delta_{ef})\{(\boldsymbol{\Omega}_{\Sigma} \odot \hat{\boldsymbol{\Sigma}}_{\mathrm{ML}(-i)})^{-1}(\mathbf{y}_i - \boldsymbol{\Omega}_{\mathrm{B}} \odot \hat{\mathbf{B}}_{\mathrm{ML}(-i)}\mathbf{x}_i)\}_e \\ &\times \{(\boldsymbol{\Omega}_{\Sigma} \odot \hat{\boldsymbol{\Sigma}}_{\mathrm{ML}(-i)})^{-1}(\mathbf{y}_i - \boldsymbol{\Omega}_{\mathrm{B}} \odot \hat{\mathbf{B}}_{\mathrm{ML}(-i)}\mathbf{x}_i)\}_f\,\hat{\sigma}_{\mathrm{ML}(-i)ef}\end{aligned} \tag{S3.3}$$

$$(q^* \geq e \geq f \geq 1),$$

$$\frac{\partial^2 D_{(-)}}{\partial \omega_{\mathrm{B}ab}\partial \omega_{\mathrm{B}cd}} = \sum_{i=1}^{n}\{(\boldsymbol{\Omega}_{\Sigma} \odot \hat{\boldsymbol{\Sigma}}_{\mathrm{ML}(-i)})^{-1}\}_{ac}(\mathbf{x}_i\mathbf{x}_i')_{db}(\hat{\mathbf{B}}_{\mathrm{ML}(-i)})_{ab}(\hat{\mathbf{B}}_{\mathrm{ML}(-i)})_{cd}$$

$$(a, c = 1,\ldots,q^*;\ b, d = 1,\ldots,p^*),$$

$$\frac{\partial^2 D_{(-)}}{\partial\omega_{\mathrm{B}ab}\partial\omega_{\Sigma ef}} = \sum_{i=1}^{n}\sum_{(e,f)}^{2}\frac{1}{2}(2-\delta_{ef})\{(\mathbf{\Omega}_\Sigma \odot \hat{\mathbf{\Sigma}}_{\mathrm{ML}(-i)})^{-1}\}_{ae}$$

$$\times\{(\mathbf{\Omega}_\Sigma \odot \hat{\mathbf{\Sigma}}_{\mathrm{ML}(-i)})^{-1}(\mathbf{y}_i - \mathbf{\Omega}_\mathrm{B} \odot \hat{\mathbf{B}}_{\mathrm{ML}(-i)}\mathbf{x}_i)\mathbf{x}_i'\}_{fb}(\hat{\mathbf{B}}_{\mathrm{ML}(-i)})_{ab}\hat{\sigma}_{\mathrm{ML}(-i)ef}$$

$$(a=1,...,q^*;\ b=1,...,p^*;\ q^* \ge e \ge f \ge 1),$$

$$\frac{\partial^2 D_{(-)}}{\partial\omega_{\Sigma ef}\partial\omega_{\Sigma gh}} = -\sum_{i=1}^{n}\frac{1}{4}(2-\delta_{ef})(2-\delta_{gh})$$

$$\times[\{(\mathbf{\Omega}_\Sigma \odot \hat{\mathbf{\Sigma}}_{\mathrm{ML}(-i)})^{-1}\}_{eg}\{(\mathbf{\Omega}_\Sigma \odot \hat{\mathbf{\Sigma}}_{\mathrm{ML}(-i)})^{-1}\}_{hf}$$

$$+\{(\mathbf{\Omega}_\Sigma \odot \hat{\mathbf{\Sigma}}_{\mathrm{ML}(-i)})^{-1}\}_{eh}\{(\mathbf{\Omega}_\Sigma \odot \hat{\mathbf{\Sigma}}_{\mathrm{ML}(-i)})^{-1}\}_{gf}]\hat{\sigma}_{\mathrm{ML}(-i)ef}\hat{\sigma}_{\mathrm{ML}(-i)gh}$$

$$+\sum_{i=1}^{n}\frac{1}{4}(2-\delta_{ef})(2-\delta_{gh})\sum_{(e,f)}^{2}\sum_{(g,h)}^{2}\{(\mathbf{\Omega}_\Sigma \odot \hat{\mathbf{\Sigma}}_{\mathrm{ML}(-i)})^{-1}\}_{eg}$$

$$\times\{(\mathbf{\Omega}_\Sigma \odot \hat{\mathbf{\Sigma}}_{\mathrm{ML}(-i)})^{-1}(\mathbf{y}_i - \mathbf{\Omega}_\mathrm{B} \odot \hat{\mathbf{B}}_{\mathrm{ML}(-i)}\mathbf{x}_i)\}_h$$

$$\times\{(\mathbf{\Omega}_\Sigma \odot \hat{\mathbf{\Sigma}}_{\mathrm{ML}(-i)})^{-1}(\mathbf{y}_i - \mathbf{\Omega}_\mathrm{B} \odot \hat{\mathbf{B}}_{\mathrm{ML}(-i)}\mathbf{x}_i)\}_f\hat{\sigma}_{\mathrm{ML}(-i)ef}\hat{\sigma}_{\mathrm{ML}(-i)gh}$$

$$(q^* \ge e \ge f \ge 1; q^* \ge g \ge h \ge 1).$$

S3.2 Multivariate multiple regression with normal errors when the covariance matrix of errors is given

$$\boldsymbol{\omega} = \mathrm{vec}(\mathbf{\Omega}_\mathrm{B}),$$

$$L_{(-)} \equiv \prod_{i=1}^{n}\frac{1}{(2\pi)^{q^*/2}|\mathbf{\Sigma}|^{1/2}}\exp\{-(\mathbf{y}_i - \mathbf{\Omega}_\mathrm{B} \odot \hat{\mathbf{B}}_{\mathrm{ML}(-i)}\mathbf{x}_i)'$$

$$\times\mathbf{\Sigma}^{-1}(\mathbf{y}_i - \mathbf{\Omega}_\mathrm{B} \odot \hat{\mathbf{B}}_{\mathrm{ML}(-i)}\mathbf{x}_i)/2\},$$

$$D_{(-)} = -\ln L_{(-)} = \frac{nq^*}{2}\ln(2\pi) + \frac{n}{2}\ln|\mathbf{\Sigma}|$$

$$+\frac{1}{2}\sum_{i=1}^{n}(\mathbf{y}_i - \mathbf{\Omega}_\mathrm{B} \odot \hat{\mathbf{B}}_{\mathrm{ML}(-i)}\mathbf{x}_i)'\mathbf{\Sigma}^{-1}(\mathbf{y}_i - \mathbf{\Omega}_\mathrm{B} \odot \hat{\mathbf{B}}_{\mathrm{ML}(-i)}\mathbf{x}_i), \qquad \text{(S3.4)}$$

$$\frac{\partial D_{(-)}}{\partial\mathbf{\Omega}_\mathrm{B}} = -\sum_{i=1}^{n}\{\mathbf{\Sigma}^{-1}(\mathbf{y}_i - \mathbf{\Omega}_\mathrm{B} \odot \hat{\mathbf{B}}_{\mathrm{ML}(-i)}\mathbf{x}_i)\mathbf{x}_i'\} \odot \hat{\mathbf{B}}_{\mathrm{ML}(-i)},$$

giving

$$\sum_{i=1}^{n}\{\mathbf{\Sigma}^{-1}(\mathbf{\Omega}_{\mathrm{B}}\odot\hat{\mathbf{B}}_{\mathrm{ML}(-i)})\mathbf{x}_i\mathbf{x}_i'\}\odot\hat{\mathbf{B}}_{\mathrm{ML}(-i)}=\sum_{i=1}^{n}(\mathbf{\Sigma}^{-1}\mathbf{y}_i\mathbf{x}_i')\odot\hat{\mathbf{B}}_{\mathrm{ML}(-i)}, \quad \text{(S3.5)}$$

which yields a simultaneous linear equation as $\mathbf{A}\mathrm{vec}(\mathbf{\Omega}_{\mathrm{B}})=\mathbf{a}$, where

$$(\mathbf{A})_{(ab,cd)}=(\mathbf{\Sigma}^{-1})_{ac}\sum_{i=1}^{n}(\hat{\mathbf{B}}_{\mathrm{ML}(-i)})_{cd}(\mathbf{x}_i\mathbf{x}_i')_{db}(\hat{\mathbf{B}}_{\mathrm{ML}(-i)})_{ab},$$

$$(\mathbf{a})_{(ab)}=\sum_{i=1}^{n}(\mathbf{\Sigma}^{-1}\mathbf{y}_i\mathbf{x}_i')_{ab}(\hat{\mathbf{B}}_{\mathrm{ML}(-i)})_{ab} \quad \text{(S3.6)}$$

$$(a,c=1,\ldots,q^*;b,d=1,\ldots,p^*)$$

where a double subscript notation for rows and columns is used. From (S3.6), $\mathrm{vec}(\hat{\mathbf{\Omega}}_{\mathrm{B}})=\mathbf{A}^{-1}\mathbf{a}$ is obtained.

S3.3 Multivariate multiple regression for non-normal errors or normal errors with $\mathbf{\Sigma}=k\mathbf{I}_{(q*)}$, where k is a positive constant and $\mathbf{I}_{(q*)}$ is the $q^*\times q^*$ identity matrix

$$\boldsymbol{\omega}=\mathrm{vec}(\mathbf{\Omega}_{\mathrm{B}}),$$

$$D_{(-)}=\sum_{i=1}^{n}(\mathbf{y}_i-\mathbf{\Omega}_{\mathrm{B}}\odot\hat{\mathbf{B}}_{\mathrm{ML}(-i)}\mathbf{x}_i)'(\mathbf{y}_i-\mathbf{\Omega}_{\mathrm{B}}\odot\hat{\mathbf{B}}_{\mathrm{ML}(-i)}\mathbf{x}_i), \quad \text{(S3.7)}$$

where

$$\hat{\mathbf{B}}_{(-i)}=\hat{\mathbf{B}}_{\mathrm{ML}(-i)}=(\mathbf{X}_{(-i)}'\mathbf{X}_{(-i)})^{-1}\mathbf{X}_{(-i)}'\mathbf{Y}_{(-i)}(i=1,\ldots,n) \quad \text{(S3.8)}$$

is used as before,

$$\frac{\partial D_{(-)}}{\partial\mathbf{\Omega}_{\mathrm{B}}}=-2\sum_{i=1}^{n}\{(\mathbf{y}_i-\mathbf{\Omega}_{\mathrm{B}}\odot\hat{\mathbf{B}}_{(-i)}\mathbf{x}_i)\mathbf{x}_i'\}\odot\hat{\mathbf{B}}_{(-i)}. \quad \text{(S3.9)}$$

Then, we obtain

$$\sum_{i=1}^{n}\{(\mathbf{\Omega}_{\mathrm{B}}\odot\hat{\mathbf{B}}_{(-i)})\mathbf{x}_i\mathbf{x}_i'\}\odot\hat{\mathbf{B}}_{(-i)}=\sum_{i=1}^{n}(\mathbf{y}_i\mathbf{x}_i')\odot\hat{\mathbf{B}}_{(-i)}, \quad \text{(S3.10)}$$

which yields a simultaneous linear equation $\mathbf{A}\mathrm{vec}(\mathbf{\Omega}_{\mathrm{B}})=\mathbf{a}$, where

$$(\mathbf{A})_{(ab,ac)}=\sum_{i=1}^{n}(\hat{\mathbf{B}}_{(-i)})_{ac}(\mathbf{x}_i\mathbf{x}_i')_{cb}(\hat{\mathbf{B}}_{(-i)})_{ab},$$

$$(\mathbf{a})_{(ab)}=\sum_{i=1}^{n}(\mathbf{y}_i\mathbf{x}_i')_{ab}(\hat{\mathbf{B}}_{(-i)})_{ab}\ (a=1,\ldots,q^*;b,c=1,\ldots,p^*), \quad \text{(S3.11)}$$

which gives $\mathrm{vec}(\hat{\mathbf{\Omega}}_{\mathrm{B}}) = \mathbf{A}^{-1}\mathbf{a}$.

S4. Poisson regression under canonical parametrization

$$L_{(-)} = \prod_{i=1}^{n} \exp\{-\exp(\mathbf{x}_i{}'\boldsymbol{\omega} \odot \hat{\boldsymbol{\beta}}_{\mathrm{ML}(-i)})\} \exp(\mathbf{x}_i{}'\boldsymbol{\omega} \odot \hat{\boldsymbol{\beta}}_{\mathrm{ML}(-i)} y_i) / y_i!,$$

$(y_i = 0, 1, 2 \ldots)$,

$$D_{(-)} = \sum_{i=1}^{n} \{\exp(\mathbf{x}_i{}'\boldsymbol{\omega} \odot \hat{\boldsymbol{\beta}}_{\mathrm{ML}(-i)}) - \mathbf{x}_i{}'\boldsymbol{\omega} \odot \hat{\boldsymbol{\beta}}_{\mathrm{ML}(-i)} y_i + \ln(y_i!)\},$$

$$\frac{\partial D_{(-)}}{\partial \boldsymbol{\omega}} = \sum_{i=1}^{n} \{\exp(\mathbf{x}_i{}'\boldsymbol{\omega} \odot \hat{\boldsymbol{\beta}}_{\mathrm{ML}(-i)}) - y_i\} \mathbf{x}_i \odot \hat{\boldsymbol{\beta}}_{\mathrm{ML}(-i)}, \tag{S4.1}$$

$$\frac{\partial^2 D_{(-)}}{\partial \boldsymbol{\omega} \partial \boldsymbol{\omega}'} = \sum_{i=1}^{n} \exp(\mathbf{x}_i{}'\boldsymbol{\omega} \odot \hat{\boldsymbol{\beta}}_{\mathrm{ML}(-i)}) (\mathbf{x}_i \odot \hat{\boldsymbol{\beta}}_{\mathrm{ML}(-i)})(\mathbf{x}_i \odot \hat{\boldsymbol{\beta}}_{\mathrm{ML}(-i)})'.$$

S5. Logistic regression

$$L_{(-)} = \prod_{i=1}^{n} \left\{ \frac{1}{1 + \exp(-\mathbf{x}_i{}'\boldsymbol{\omega} \odot \hat{\boldsymbol{\beta}}_{\mathrm{ML}(-i)})} \right\}^{y_i} \left\{ \frac{\exp(-\mathbf{x}_i{}'\boldsymbol{\omega} \odot \hat{\boldsymbol{\beta}}_{\mathrm{ML}(-i)})}{1 + \mathrm{xp}(-\mathbf{x}_i{}'\boldsymbol{\omega} \odot \hat{\boldsymbol{\beta}}_{\mathrm{ML}(-i)})} \right\}^{1-y_i}$$

$$= \prod_{i=1}^{n} \frac{\exp\{-(1-y_i)\mathbf{x}_i{}'\boldsymbol{\omega} \odot \hat{\boldsymbol{\beta}}_{\mathrm{ML}(-i)}\}}{1 + \exp(-\mathbf{x}_i{}'\boldsymbol{\omega} \odot \hat{\boldsymbol{\beta}}_{\mathrm{ML}(-i)})}, \quad (y_i = 0, 1),$$

$$D_{(-)} = \sum_{i=1}^{n} \left[(1-y_i)\mathbf{x}_i{}'\boldsymbol{\omega} \odot \hat{\boldsymbol{\beta}}_{\mathrm{ML}(-i)} + \ln\{1 + \exp(-\mathbf{x}_i{}'\boldsymbol{\omega} \odot \hat{\boldsymbol{\beta}}_{\mathrm{ML}(-i)})\} \right],$$

$$\frac{\partial D_{(-)}}{\partial \boldsymbol{\omega}} = \sum_{i=1}^{n} \left\{ 1 - y_i - \frac{\exp(-\mathbf{x}_i{}'\boldsymbol{\omega} \odot \hat{\boldsymbol{\beta}}_{\mathrm{ML}(-i)})}{1 + \exp(-\mathbf{x}_i{}'\boldsymbol{\omega} \odot \hat{\boldsymbol{\beta}}_{\mathrm{ML}(-i)})} \right\} \mathbf{x}_i \odot \hat{\boldsymbol{\beta}}_{\mathrm{ML}(-i)}$$

$$= \sum_{i=1}^{n} \left\{ \frac{1}{1 + \exp(-\mathbf{x}_i{}'\boldsymbol{\omega} \odot \hat{\boldsymbol{\beta}}_{\mathrm{ML}(-i)})} - y_i \right\} \mathbf{x}_i \odot \hat{\boldsymbol{\beta}}_{\mathrm{ML}(-i)},$$

$$\frac{\partial^2 D_{(-)}}{\partial \boldsymbol{\omega} \partial \boldsymbol{\omega}'} = \sum_{i=1}^{n} P_{(-i)}^{(\omega)} Q_{(-i)}^{(\omega)} (\mathbf{x}_i \odot \hat{\boldsymbol{\beta}}_{\mathrm{ML}(-i)})(\mathbf{x}_i \odot \hat{\boldsymbol{\beta}}_{\mathrm{ML}(-i)})',$$

$$P_{(-i)}^{(\omega)} = \frac{1}{1+\exp(-\mathbf{x}_i{}'\boldsymbol{\omega} \odot \hat{\boldsymbol{\beta}}_{\mathrm{ML}(-i)})}, Q_{(-i)}^{(\omega)} = 1 - P_{(-i)}^{(\omega)} \ (i=1,...,n). \qquad \text{(S5.1)}$$

Reference

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