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Supplement III to the paper "Asymptotic cumulants of some information criteria" – Examples 2 and 3

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This article is to supplement Ogasawara (2016) with Examples 2 and 3 using expository computations.

S6. Example 2: The normal distribution with the MLE of the mean and the known variance σ^2 under possible non-normality

S6.1 Preliminary results

$$f(x^* = x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}.$$

For $j = 1,\ldots,n,$

$$l_j = -\frac{1}{2}\log(2\pi\sigma^2) - \frac{(x_j-\mu)^2}{2\sigma^2},\ \ \bar{l} = n^{-1}\sum_{k=1}^{n} l_k,$$

$$l_{0j} = -\frac{1}{2}\log(2\pi\sigma^2) - \frac{(x_j-\mu_0)^2}{2\sigma^2},\ \ \bar{l}_0 = n^{-1}\sum_{k=1}^{n} l_{0k},$$

$$\bar{l}_0^* = \mathrm{E}_g(l_{0j}) = \mathrm{E}_f(l_{0j}) = \mathrm{E}_g(\bar{l}_0) = \mathrm{E}_f(\bar{l}_0) = -\frac{1}{2}\log(2\pi\sigma^2) - \frac{1}{2},$$

$$\theta = \mu,\ \theta_0 = \mu_0,\ \hat{\theta}_{\mathrm{ML}} = \hat{\mu} = \bar{x},\ n\,\mathrm{var}_g(\hat{\theta}_{\mathrm{ML}}) = n\,\mathrm{var}_f(\hat{\theta}_{\mathrm{ML}}) = \sigma^2,$$

$$\frac{\partial\bar{l}}{\partial\theta_0} = \frac{\bar{x}-\mu_0}{\sigma^2},\ \frac{\partial l_j}{\partial\theta_0} = \frac{x_j-\mu_0}{\sigma^2},\ \boldsymbol{\Lambda} = \lambda = \frac{\partial^2\bar{l}}{\partial\theta_0^2} = \frac{\partial^2 l_j}{\partial\theta_0^2} = -\frac{1}{\sigma^2} = -\bar{i}_0,$$

$$\boldsymbol{\Gamma} = \gamma = \mathrm{E}_g\left\{\left(\frac{\partial l_j}{\partial\theta_0}\right)^2\right\} = n\mathrm{E}_g\left\{\left(\frac{\partial\bar{l}}{\partial\theta_0}\right)^2\right\} = n\mathrm{E}_f\left\{\left(\frac{\partial\bar{l}}{\partial\theta_0}\right)^2\right\} = \frac{1}{\sigma^2} = \bar{i}_0, \tag{S6.1}$$

$$\mathrm{tr}(-\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}) = -\lambda^{-1}\gamma = 1,\ \ l_{0j} - \overline{l}_0^{\,*} = -\frac{1}{2\sigma^2}\{(x_j - \mu_0)^2 - \sigma^2\} \equiv -\frac{z_j^2 - 1}{2}$$

(note that $\lambda = -\gamma$ even under non-normality),

$$\frac{\partial l_j}{\partial \theta_0} = \frac{z_j}{\sigma},\ \mathbf{J}_0^{(3)} = j_0^{(3)} = \frac{\partial^3 \overline{l}}{\partial \theta_0^3} = 0,\ \mathbf{J}_0^{(4)} = j_0^{(4)} = \frac{\partial^4 \overline{l}}{\partial \theta_0^4} = 0.$$

Let $\kappa_k \equiv \kappa_{gk}(z)\ (k = 1, 2, ...)$ under possible non-normality with $z = (x^* - \mu_0)/\sigma$. Note that the notation κ_k is used only when the argument of $\kappa_{gk}(\cdot)$ is *z* for simplicity.

$$v_0^{(\mathrm{A})} = 4(n-1)^{-1}\sum_{j=1}^{n}(l_{0j} - \overline{l}_0)^2 = 2(n^2 - n)^{-1}\sum_{j,k=1}^{n}(l_{0j} - l_{0k})^2 = O_p(1),$$

$$\hat{v}_{\mathrm{ML}}^{(\mathrm{A})} = 4(n-1)^{-1}\sum_{j=1}^{n}(\hat{l}_{\mathrm{ML}j} - \hat{\overline{l}}_{\mathrm{ML}})^2 = \hat{\alpha}_{\mathrm{ML}2}^{(\mathrm{A})} = O_p(1),$$

$$\hat{l}_{\mathrm{ML}j} = l_j\,|_{\theta = \hat{\theta}_{\mathrm{ML}}},\ \hat{\overline{l}}_{\mathrm{ML}} = n^{-1}\sum_{j=1}^{n}\hat{l}_{\mathrm{ML}j},$$

$$\kappa_{g2}(l_{0j}) = \mathrm{var}_g(l_{0j}) = \frac{1}{4}\mathrm{var}_g(z^2) = \frac{1}{4}(\kappa_4 + 2),$$

$$\kappa_{g3}(l_{0j}) = -\frac{1}{8}\kappa_{g3}(z^2) = -\frac{1}{8}\mathrm{E}_g\{(z^2-1)^3\}$$

$$= -\frac{1}{8}(\kappa_6 + 12\kappa_4 + 10\kappa_3^2 + 8)$$

(see (S6.8) given later), (S6.2)

$$\kappa_{g4}(l_{0j}) = \frac{1}{16}\kappa_{g4}(z^2)$$

$$= \frac{1}{16}(\kappa_8 + 24\kappa_6 + 56\kappa_5\kappa_3 + 32\kappa_4^2 + 144\kappa_4 + 240\kappa_3^2 + 48)$$

(see also (S6.10) given later),

$$\mathrm{E}_{\mathrm{g}}\left(\frac{\partial l_j}{\partial \theta_0} l_{0j}\right) = \mathrm{E}_{\mathrm{g}}\left\{\frac{z}{\sigma}\left(-\frac{z^2}{2}\right)\right\} = -\frac{\kappa_3}{2\sigma},$$

$$\mathrm{E}_{\mathrm{g}}\{(l_{0j} - \overline{l}_0^*)^4\} = \frac{1}{16}\mathrm{E}_{\mathrm{g}}\{(z^2-1)^4\} = \frac{1}{16}\mathrm{E}_{\mathrm{g}}(z^8 - 4z^6 + 6z^4 - 4z^2 + 1)$$

$$= \frac{1}{16}\{\kappa_8 + 28\kappa_6 + 56\kappa_5\kappa_3 + 35\kappa_4^2 + 210\kappa_4 + 280\kappa_3^2 + 105$$

$$-4(\kappa_6 + 15\kappa_4 + 10\kappa_3^2 + 15) + 6(\kappa_4 + 3) - 4 + 1\}$$

$$= \frac{1}{16}(\kappa_8 + 24\kappa_6 + 56\kappa_5\kappa_3 + 35\kappa_4^2 + 156\kappa_4 + 240\kappa_3^2 + 60)$$

(see also (S6.10)),

$$\mathrm{E}_{\mathrm{g}}\left\{\frac{\partial l_j}{\partial \theta_0}(l_{0j} - \overline{l}_0^*)^2\right\} = \mathrm{E}_{\mathrm{g}}\left\{\frac{z}{\sigma}\left(-\frac{z^2-1}{2}\right)^2\right\} = \frac{1}{4\sigma}\mathrm{E}_{\mathrm{g}}(z^5 - 2z^3 + z)$$

$$= \frac{1}{4\sigma}(\kappa_5 + 10\kappa_3 - 2\kappa_3) = \frac{1}{4\sigma}(\kappa_5 + 8\kappa_3),$$

$$\mathrm{E}_{\mathrm{g}}\left\{(l_{0j} - \overline{l}_0^*)\left(\frac{\partial l_j}{\partial \theta_0}\right)^2\right\} = \mathrm{E}_{\mathrm{g}}\left(-\frac{z^2-1}{2}\frac{z^2}{\sigma^2}\right) = -\frac{1}{2\sigma^2}\mathrm{E}_{\mathrm{g}}(z^4 - z^2)$$

$$= -\frac{1}{2\sigma^2}(\kappa_4 + 2),$$

$$\mathrm{E}_g(v_0^{(\mathrm{A})}) = \alpha_{\mathrm{ML2}}^{(\mathrm{A})} = \mathrm{var}_g(-2l_{0j}) = \kappa_4 + 2,$$

$$\mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial \theta_0}\right) = 8\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial \theta_0}\right) = -\frac{4\kappa_3}{\sigma},\ \mathrm{E}_g\left(\frac{\partial^2 v^{(\mathrm{A})}}{\partial \theta_0^2}\right) = 8\gamma = \frac{8}{\sigma^2}$$

(the result 8γ holds under canonical parametrization),

$$\mathrm{E}_g\left\{\left(\frac{\partial v^{(\mathrm{A})}}{\partial\theta_0}\right)^2\right\}=64\left\{\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial\theta_0}\right)\right\}^2+O(n^{-1})=\frac{16\kappa_3^2}{\sigma^2}+O(n^{-1}),$$

$$n\,\mathrm{cov}_g(m_v,\overline{l}_0)=n\,\mathrm{cov}_g(v_0^{(\mathrm{A})},\overline{l}_0)=4\mathrm{E}_g\{(l_{0j}-\overline{l}_0^{*})^3\}=4\kappa_{g3}(l_{0j})$$

$$=-\frac{1}{2}(\kappa_6+12\kappa_4+10\kappa_3^2+8),$$

$$n\,\mathrm{cov}_g\left(\frac{\partial\overline{l}}{\partial\theta_0},m_v\right)=4\mathrm{E}_g\left\{\frac{\partial l_j}{\partial\theta_0}(l_{0j}-\overline{l}_0^{*})^2\right\}=\frac{1}{\sigma}(\kappa_5+8\kappa_3),$$

$$n\,\mathrm{avar}_g(m_v)=16\Big[\mathrm{E}_g\{(l_{0j}-\overline{l}_0^{*})^4\}-[\mathrm{E}_g\{(l_{0j}-\overline{l}_0^{*})^2\}]^2\Big]$$

$$=16\left[\frac{1}{16}\mathrm{E}_g\{(z^2-1)^4\}-\left\{\frac{1}{4}\mathrm{E}_g\{(z^2-1)^2\}\right\}^2\right]$$

$$=\mathrm{E}_g\{(z^2-1)^4\}-[\mathrm{E}_g\{(z^2-1)^2\}]^2$$

$$=\kappa_8+24\kappa_6+56\kappa_5\kappa_3+35\kappa_4^2+156\kappa_4+240\kappa_3^2+60-(\kappa_4+2)^2$$

$$=\kappa_8+24\kappa_6+56\kappa_5\kappa_3+34\kappa_4^2+152\kappa_4+240\kappa_3^2+56,$$

$$n\,\mathrm{cov}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial\theta_0},\frac{\partial\overline{l}_0}{\partial\theta_0}\right)=8\mathrm{E}_g\left\{(l_{0j}-\overline{l}_0^{*})\left(\frac{\partial l_j}{\partial\theta_0}\right)^2\right\}=-\frac{4}{\sigma^2}(\kappa_4+2),$$

$$n\,\mathrm{cov}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial\theta_0},\overline{l}_0\right)=8\mathrm{E}_g\left\{(l_{0j}-\overline{l}_0^{*})^2\frac{\partial l_j}{\partial\theta_0}\right\}=\frac{2}{\sigma}(\kappa_5+8\kappa_3),$$

$$\mathbf{m}_v^{(1)}=\left(m_v,\frac{\partial\overline{l}}{\partial\theta_0}\right)',$$

$$\mathbf{m}_v^{(2)}=\left[m_v^2,\ m_v\frac{\partial\overline{l}}{\partial\theta_0},\ 0,\ \left(\frac{\partial\overline{l}}{\partial\theta_0}\right)^2,\ \left\{\frac{\partial v^{(\mathrm{A})}}{\partial\theta_0}-\mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial\theta_0}\right)\right\}\otimes\frac{\partial\overline{l}}{\partial\theta_0}\right]',$$

$$\mathbf{v}^{(1)} = \frac{1}{2}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\left\{-1,\ \mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial\theta_0}\right)\lambda^{-1}\right\}'$$

$$= \frac{1}{2}(\kappa_4+2)^{-3/2}\left\{-1, -\frac{4\kappa_3}{\sigma}\left(-\frac{1}{\sigma^2}\right)^{-1}\right\}' = \frac{1}{2}(\kappa_4+2)^{-3/2}(-1,\ 4\sigma\kappa_3)'$$

(under normality $\mathbf{v}^{(1)} = (-2^{-5/2}, 0)'$),

$$\mathbf{v}^{(2)} = \underset{(\mathrm{A})}{\Bigg[}\ \frac{3}{8}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-5/2},\ -\frac{3}{4}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-5/2}\mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial\theta_0}\right)\lambda^{-1},$$

$$-\frac{1}{2}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\mathrm{E}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial\theta_0}\right)(\lambda^{-1})^2,\ \underset{(\mathrm{B})}{\Bigg[}\ -\frac{1}{4}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\mathrm{E}_g\left(\frac{\partial^2 v^{(\mathrm{A})}}{\partial\theta_0^2}\right)$$

$$+\frac{3}{8}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-5/2}\mathrm{E}_g\left\{\left(\frac{\partial v^{(\mathrm{A})}}{\partial\theta_0}\right)^2\right\} + O(n^{-1})\ \underset{(\mathrm{B})}{\Bigg]}(\lambda^{-1})^2, \frac{1}{2}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\lambda^{-1}\ \underset{(\mathrm{A})}{\Bigg]}'$$

$$= \left[\frac{3}{8}(\kappa_4+2)^{-5/2}, -3(\kappa_4+2)^{-5/2}\kappa_3\sigma,\ 2(\kappa_4+2)^{-3/2}\kappa_3\sigma^3,\right.$$

$$\left.\{-2(\kappa_4+2)^{-3/2} + 6(\kappa_4+2)^{-5/2}\kappa_3^2\}\sigma^2, -\frac{\sigma^2}{2}(\kappa_4+2)^{-3/2}\right]'$$

(under normality $\mathbf{v}^{(2)} = (3\times 2^{-11/2},\ 0,\ 0,\ -2^{-1/2}\sigma^2, -2^{-5/2}\sigma^2)'$),

$$n^2\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^2\mathbf{m}_v^{(1)}{}'\} = 4\left[4\kappa_{g4}(l_{0j}),\ \mathrm{E}_g\left\{(l_{0j}-\bar{l}_0^*)^2\frac{\partial l_j}{\partial\theta_0}\right\}\right]$$

$$= \left\{16\kappa_{g4}(l_{0j}),\ \frac{1}{\sigma}(\kappa_5+8\kappa_3)\right\}$$

(under normality $n^2\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^2\mathbf{m}_v^{(1)}{}'\} = (48,\ 0)$),

$$n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)} = -2\left\{n\,\mathrm{cov}_g(\bar{l}_0, m_v),\ n\,\mathrm{cov}_g\left(\bar{l}_0, \frac{\partial \bar{l}}{\partial \theta_0}\right)\right\}\mathbf{v}^{(1)}$$

$$= -2\left\{4\kappa_{g3}(l_{0j}), -\frac{\kappa_3}{2\sigma}\right\}\frac{1}{2}(\kappa_4+2)^{-3/2}(-1,\ 4\sigma\kappa_3)'$$

$$= -(\kappa_4+2)^{-3/2}\{-4\kappa_{g3}(l_{0j}) - 2\kappa_3^2\}$$

$$= -(\kappa_4+2)^{-3/2}\left\{\frac{1}{2}(\kappa_6+12\kappa_4+10\kappa_3^2+8) - 2\kappa_3^2\right\}$$

$$= -\frac{1}{2}(\kappa_4+2)^{-3/2}(\kappa_6+12\kappa_4+6\kappa_3^2+8)$$

(under normality $n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)} = -2^{1/2}$),

$$-2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{ML}} - \hat{\bar{l}}_{\mathrm{ML}}^*) = n^{-1}2\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma}) + n^{-2}(c_1+c_2+c_3) + O(n^{-3})$$

$$= n^{-1}b_1 + n^{-2}b_2 + O(n^{-3}) = -n^{-1}2 \quad (b_2 = 0,\ O(n^{-3}) = 0),$$

$b_1 = -2$ even under non-normality, and $b_2 = 0$ comes from $c_2 = c_3 = 0$ under canonical parametrization and $c_1 = 0$ due to $j_0^{(3)} = j_0^{(4)} = 0$ ($O(n^{-3}) = 0$ stems from $j_0^{(k)} = 0,\ k = 5, 6, \ldots$ defined similarly to $j_0^{(k)},\ k = 3,\ 4$),

$$\bar{l}_{\mathrm{ML}}^{(1)} = -2(\bar{l}_0 - \bar{l}_0^*) = \frac{1}{\sigma^2}\left\{n^{-1}\sum_{j=1}^{n}(x_j - \mu_0)^2 - \sigma^2\right\} \equiv \overline{z^2} - 1,$$

$$\bar{l}_{\mathrm{ML}}^{(2)} = \frac{\partial \bar{l}}{\partial \theta_0}\lambda^{-1}\frac{\partial \bar{l}}{\partial \theta_0} = \left(-\frac{1}{\sigma^2}\right)^{-1}\left(\frac{\bar{x} - \mu_0}{\sigma^2}\right)^2 = -\frac{(\bar{x} - \mu_0)^2}{\sigma^2} \equiv -\bar{z}^2 \qquad \text{(S6.3)}$$

$$(\bar{z} \equiv (\bar{x} - \mu_0)/\sigma), \quad n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(2)}) = \lambda^{-1}\gamma = -1, \quad \bar{l}_{\mathrm{ML}}^{(3)} = \bar{l}_{\mathrm{ML}}^{(4)} = 0,$$

where $\bar{l}_{\mathrm{ML}}^{(3)} = \bar{l}_{\mathrm{ML}}^{(4)} = 0$ is due to $j_0^{(3)} = j_0^{(4)} = 0$.

S6.2 $n^{-1}\mathrm{AIC}_{\mathrm{ML}}$

$$n^{-1}\mathrm{AIC}_{\mathrm{ML}} = -2\hat{\bar{l}}_{\mathrm{ML}} + n^{-1}2q = -2\hat{\bar{l}}_{\mathrm{ML}} + n^{-1}2\,.$$

S6.2.1 Asymptotic cumulants of $n^{-1}\mathrm{AIC}_{\mathrm{ML}}$ before studentization

For estimation of $-2\mathrm{E}_g(\hat{\bar{l}}^*_{\mathrm{ML}})$,

$$\kappa_{g1}\{n^{-1}\mathrm{AIC}_{\mathrm{ML}} + 2\mathrm{E}_g(\hat{\bar{l}}^*_{\mathrm{ML}})\}$$

$$= -2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{ML}} - \hat{\bar{l}}^*_{\mathrm{ML}}) + n^{-1}2q = -n^{-1}2 + n^{-1}2 = 0 \qquad \text{(S6.4)}$$

(exactly 0; $\alpha^{(\mathrm{A})*}_{\mathrm{ML1}} = \alpha^{(\mathrm{A})*}_{\mathrm{ML\Delta 1}} = 0$)

while for estimation of $-2\bar{l}_0^*$,

$$\kappa_{g1}(n^{-1}\mathrm{AIC}_{\mathrm{ML}} + 2\bar{l}_0^*) = n^{-1}(2q + \lambda^{-1}\gamma) + n^{-2}\{n^2\mathrm{E}_g(\bar{l}^{(3)}_{\mathrm{ML}} + \bar{l}^{(4)}_{\mathrm{ML}})\} + \ldots$$

$$= n^{-1}(2-1) + n^{-2}\times 0 + n^{-3}\times 0 + \ldots = n^{-1} \qquad \text{(S6.5)}$$

$$(\alpha^{(\mathrm{A})}_{\mathrm{ML1}} = 1,\ \alpha^{(\mathrm{A})}_{\mathrm{ML\Delta 1}} = 0).$$

Using $\bar{l}^{(3)}_{\mathrm{ML}} = 0$,

$$\kappa_{g2}(n^{-1}\mathrm{AIC}_{\mathrm{ML}}) = n^{-1}[n\mathrm{E}_g\{(\bar{l}^{(1)}_{\mathrm{ML}})^2\}] + n^{-2}[2n^2\mathrm{E}_g(\bar{l}^{(1)}_{\mathrm{ML}}\bar{l}^{(2)}_{\mathrm{ML}})$$

$$+ n^2\mathrm{E}_g\{(\bar{l}^{(2)}_{\mathrm{ML}})^2\} - \{n\mathrm{E}_g(\bar{l}^{(2)}_{\mathrm{ML}})\}^2] + O(n^{-3})$$

$$= n^{-1}\frac{1}{\sigma^4}n\mathrm{E}_g\left[\left\{n^{-1}\sum_{j=1}^{n}(x_j - \mu_0)^2 - \sigma^2\right\}^2\right]$$

$$+n^{-2}\underset{(\mathrm{A})}{\Bigg[}\ -\frac{2n^2}{\sigma^4}\mathrm{E}_g\left[\left\{n^{-1}\sum_{j=1}^{n}(x_j - \mu_0)^2 - \sigma^2\right\}(\bar{x} - \mu_0)^2\right] \qquad \text{(S6.6)}$$

$$+\frac{n^2}{\sigma^4}\mathrm{E}_g\{(\bar{x} - \mu_0)^4\} - (-1)^2\ \underset{(\mathrm{A})}{\Bigg]} + O(n^{-3}),$$

where the first term on the right-hand side of (S6.6) is

$$n^{-1}[n\mathrm{E}_g\{(\bar{l}^{(1)}_{\mathrm{ML}})^2\}] = n^{-1}[n\mathrm{E}_g\{(\overline{z^2} - 1)^2\}]$$

$$= n^{-1}[\mathrm{E}_g(z^4) - \{\mathrm{E}_g(z^2)\}^2] = n^{-1}(\kappa_4 + 3 - 1) = n^{-1}(\kappa_4 + 2),$$

the first term in $\underset{(\mathrm{A})}{[}\ \cdot\ \underset{(\mathrm{A})}{]}$ of (S6.6) is

$$2n^2\mathrm{E}_g(\overline{l}^{(1)}_{\mathrm{ML}}\overline{l}^{(2)}_{\mathrm{ML}}) = -\frac{2n^2}{\sigma^4}\mathrm{E}_g\left[\left\{n^{-1}\sum_{j=1}^{n}(x_j-\mu_0)^2-\sigma^2\right\}(\overline{x}-\mu_0)^2\right]$$

$$= -\frac{2n^2}{\sigma^4}\underset{\text{(B)}}{\Bigg[}\ \mathrm{E}_g\left[n^{-1}\sum_{j=1}^{n}(x_j-\mu_0)^2 n^{-2}\left\{\sum_{j=1}^{n}(x_j-\mu_0)\right\}^2\right]-n^{-1}\sigma^4\ \underset{\text{(B)}}{\Bigg]}$$

$$= -\frac{2n^2}{\sigma^4}\underset{\text{(B)}}{[}\ n^{-3}n\mathrm{E}_g\{(x^*-\mu_0)^4\}+n^{-3}(n^2-n)[\mathrm{E}_g\{(x^*-\mu_0)^2\}]^2-n^{-1}\sigma^4\ \underset{\text{(B)}}{]}$$

$$= -\frac{2}{\sigma^4}\{\kappa_{g4}(x^*)+3\sigma^4-\sigma^4\} = -2(\kappa_4+2),$$

the second term in $\underset{\text{(A)}}{[}\ \cdot\ \underset{\text{(A)}}{]}$ of (S6.6) is

$$n^2\mathrm{E}_g\{(\overline{l}^{(1)}_{\mathrm{ML}})^2\} = \frac{n^2}{\sigma^4}\mathrm{E}_g\{(\overline{x}-\mu_0)^4\} = 3+O(n^{-1})$$

(under normality $n^2\mathrm{E}_g\{(\overline{l}^{(1)}_{\mathrm{ML}})^2\}=3$ without the remainder term).

Consequently,

$$\begin{aligned}\kappa_{g2}(n^{-1}\mathrm{AIC}_{\mathrm{ML}}) &= n^{-1}(\kappa_4+2)+n^{-2}\{-2(\kappa_4+2)+3-1\}+O(n^{-3})\\ &= n^{-1}(\kappa_4+2)-n^{-2}2(\kappa_4+1)+O(n^{-3}) = n^{-1}\alpha^{(\mathrm{A})}_{\mathrm{ML2}}+n^{-2}\alpha^{(\mathrm{A})}_{\mathrm{ML\Delta 2}}+O(n^{-3})\end{aligned}\tag{S6.7}$$

(under normality $\alpha^{(\mathrm{A})}_{\mathrm{ML2}}=2$ and $\alpha^{(\mathrm{A})}_{\mathrm{ML\Delta 2}}=-2$).

$$\begin{aligned}\kappa_{g3}(n^{-1}\mathrm{AIC}_{\mathrm{ML}}) = n^{-2}[\ & n^2\mathrm{E}_g\{(\overline{l}^{(1)}_{\mathrm{ML}})^3\}+3n^2\mathrm{E}_g\{(\overline{l}^{(1)}_{\mathrm{ML}})^2\overline{l}^{(2)}_{\mathrm{ML}})\}\\ & -3n\mathrm{E}_g(\overline{l}^{(2)}_{\mathrm{ML}})\alpha^{(\mathrm{A})}_{\mathrm{ML2}}\ ]+O(n^{-3})\end{aligned}\tag{S6.8}$$

where the first term in $[\cdot]$ of (S6.8) is

$$n^2\mathrm{E}_g\{(\overline{l}^{(1)}_{\mathrm{ML}})^3\} = n^2\mathrm{E}_g\{(\overline{z^2}-1)^3\} = n^2\kappa_{g3}(\overline{z^2}) = \kappa_{g3}(z^2)$$

$$= \mathrm{E}_g\{(z^2-1)^3\} = \mathrm{E}_g(z^6-3z^4+3z^2-1)$$

$$= \kappa_6+15\kappa_4+10\kappa_3^2+15-3(\kappa_4+3)+3-1 = \kappa_6+12\kappa_4+10\kappa_3^2+8$$

($=\mathrm{E}_g(z^6)-3\mathrm{E}_g(z^4)+2$; Stuart & Ort, 1994, Equation (3.38)), the sum of the second and third terms in $[\cdot]$ of (S6.8) is

$$
\begin{aligned}
&3n^2\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^2\bar{l}_{\mathrm{ML}}^{(2)})\} - 3n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(2)})\alpha_{\mathrm{ML}2}^{(\mathrm{A})} \\
&= -\frac{6}{\sigma^2}[n\mathrm{E}_g\{\bar{l}_{\mathrm{ML}}^{(1)}(\bar{x}-\mu_0)\}]^2 + O(n^{-1}) \\
&= -\frac{6}{\sigma^6}\left[\mathrm{E}_g\left\{\sum_{j=1}^{n}\{(x_j-\mu_0)^2-\sigma^2\}n^{-1}\sum_{k=1}^{n}(x_k-\mu_0)\right\}\right]^2 + O(n^{-1}) \\
&= -6\kappa_3^2 + O(n^{-1}).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\kappa_{g3}(n^{-1}\mathrm{AIC}_{\mathrm{ML}}) = n^{-2}(\kappa_6 + 12\kappa_4 + 4\kappa_3^2 + 8) + O(n^{-3}) = n^{-2}\alpha_{\mathrm{ML}3}^{(\mathrm{A})} + O(n^{-3}) \\
&(= n^{-2}[\mathrm{E}_g(z^6) - 3\mathrm{E}_g(z^4) - 6\{\mathrm{E}_g(z^3)\}^2 + 2] + O(n^{-3}); \\
&\quad \text{under normality } \alpha_{\mathrm{ML}3}^{(\mathrm{A})} = 8\text{)}. \qquad \text{(S6.9)}
\end{aligned}
$$

Use $\bar{l}_{\mathrm{ML}}^{(3)} = 0$, then

$$
\begin{aligned}
\kappa_{g4}(n^{-1}\mathrm{AIC}_{\mathrm{ML}}) = n^{-3}\ &\underset{(\mathrm{A})}{[}\ n^3\kappa_{g4}(\bar{l}_{\mathrm{ML}}^{(1)}) + 4n^3\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^3\bar{l}_{\mathrm{ML}}^{(2)}\} \\
&+ 6n^3\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^2(\bar{l}_{\mathrm{ML}}^{(2)})^2\} - 4n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(2)})\alpha_{\mathrm{ML}3}^{(\mathrm{A})} \\
&- 6\alpha_{\mathrm{ML}2}^{(\mathrm{A})}\alpha_{\mathrm{ML}\Delta 2}^{(\mathrm{A})} - 6\alpha_{\mathrm{ML}2}^{(\mathrm{A})}\{n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(2)})\}^2\ \underset{(\mathrm{A})}{]} + O(n^{-4}), \qquad \text{(S6.10)}
\end{aligned}
$$

where the first term in $\underset{(\mathrm{A})}{[}\ \cdot\ \underset{(\mathrm{A})}{]}$ of (S6.10) is

$$
\begin{aligned}
&n^3\kappa_{g4}(\bar{l}_{\mathrm{ML}}^{(1)}) = n^3\kappa_{g4}(\overline{z^2}) = \kappa_{g4}(z^2) = \mathrm{E}_g\{(z^2-1)^4\} - 3[\mathrm{E}_g\{(z^2-1)^2\}]^2 \\
&= \mathrm{E}_g(z^8 - 4z^6 + 6z^4 - 4z^2 + 1) - 3\{\mathrm{E}_g(z^4 - 2z^2 + 1)\}^2 \\
&= \kappa_8 + 28\kappa_6 + 56\kappa_5\kappa_3 + 35\kappa_4^2 + 210\kappa_4 + 280\kappa_3^2 + 105 \\
&\quad - 4(\kappa_6 + 15\kappa_4 + 10\kappa_3^3 + 15) + 6(\kappa_4 + 3) - 4 + 1 - 3(\kappa_4 + 2)^2 \\
&= \kappa_8 + 24\kappa_6 + 56\kappa_5\kappa_3 + 32\kappa_4^2 + (210 - 4\times 15 + 6 - 12)\kappa_4 + 240\kappa_3^2 \\
&\quad + 105 - 4\times 15 + 6\times 3 - 4 + 1 - 12 \\
&= \kappa_8 + 24\kappa_6 + 56\kappa_5\kappa_3 + 32\kappa_4^2 + 144\kappa_4 + 240\kappa_3^2 + 48,
\end{aligned}
$$

the second term in $\underset{\text{(A)}}{[}\cdot\underset{\text{(A)}}{]}$ of (S6.10) is

$$\begin{aligned}4n^3\mathrm{E}_g\{(\overline{l}^{(1)}_{\mathrm{ML}})^3\overline{l}^{(2)}_{\mathrm{ML}}\} &= 4[\;\; n^2\mathrm{E}_g\{(\overline{l}^{(1)}_{\mathrm{ML}})^3\}n\mathrm{E}_g(\overline{l}^{(2)}_{\mathrm{ML}}) + 3\alpha^{(\mathrm{A})}_{\mathrm{ML2}}n^2\mathrm{E}_g(\overline{l}^{(1)}_{\mathrm{ML}}\overline{l}^{(2)}_{\mathrm{ML}}) \\ &\qquad - 6n\mathrm{E}_g(\overline{l}^{(1)}_{\mathrm{ML}}\overline{z})n^2\mathrm{E}_g\{(\overline{l}^{(1)}_{\mathrm{ML}})^2\overline{z}\}\;\;] + O(n^{-1})\end{aligned}$$

$$\begin{aligned}&= 4\underset{\text{(B)}}{\Bigg[}\;\; n^2\mathrm{E}_g\{(\overline{l}^{(1)}_{\mathrm{ML}})^3\}(-1) + 3(\kappa_4+2)\{-(\kappa_4+2)\} \\ &\quad -6\kappa_3\frac{n^{-1}}{\sigma^5}\mathrm{E}_g\left[\sum_{j=1}^{n}\{(x_j-\mu_0)^2-\sigma^2\}\sum_{k=1}^{n}\{(x_k-\mu_0)^2-\sigma^2\}\sum_{l^*=1}^{n}(x_{l^*}-\mu_0)\right]\underset{\text{(B)}}{\Bigg]} \\ &\quad +O(n^{-1})\end{aligned}$$

(see the first term in $\underset{\text{(A)}}{[}\cdot\underset{\text{(A)}}{]}$ of (S6.6))

$$\begin{aligned}&= 4\underset{\text{(B)}}{\Bigg[}\;\; -n^2\mathrm{E}_g\{(\overline{l}^{(1)}_{\mathrm{ML}})^3\} - 3(\kappa_4+2)^2 - 6\kappa_3\frac{n^{-1}}{\sigma^5}\mathrm{E}_g\left[\left\{\sum_{j=1}^{n}(x_j-\mu_0)^2\right\}^2\right. \\ &\qquad \left.\times\sum_{l^*=1}^{n}(x_{l^*}-\mu_0) - 2n\sigma^2\sum_{j=1}^{n}(x_j-\mu_0)^2\sum_{l^*=1}^{n}(x_{l^*}-\mu_0)\right]\underset{\text{(B)}}{\Bigg]} + O(n^{-1})\end{aligned}$$

$$\begin{aligned}&= 4[\;\; -n^2\mathrm{E}_g\{(\overline{l}^{(1)}_{\mathrm{ML}})^3\} - 3(\kappa_4+2)^2 \\ &\qquad -6\kappa_3 n^{-1}\{n\mathrm{E}_g(z^5) + 2(n^2-n)\mathrm{E}_g(z^3)\mathrm{E}_g(z^2) - 2n^2\mathrm{E}_g(z^3)\}\;\;] + O(n^{-1}) \\ &= 4[\;\; -(\kappa_6+12\kappa_4+10\kappa_3^2+8) - 3(\kappa_4^2+4\kappa_4+4) - 6\kappa_3(\kappa_5+10\kappa_3-2\kappa_3)\;\;] \\ &\qquad +O(n^{-1}) \\ &= -4(\kappa_6+24\kappa_4+58\kappa_3^2+3\kappa_4^2+6\kappa_3\kappa_5+20)+O(n^{-1})\end{aligned}$$

(see (S6.8); Stuart & Ort, 1994, Equation (3.38); note that $n^3 = n + 3(n^2-n) + n(n-1)(n-2)$).

the third term in $\underset{\text{(A)}}{[}\cdot\underset{\text{(A)}}{]}$ of (S6.10) is

$$
\begin{aligned}
&6n^{3}\mathrm{E}_{g}\{(\overline{l}_{\mathrm{ML}}^{(1)})^{2}(\overline{l}_{\mathrm{ML}}^{(2)})^{2}\}=6n^{3}\mathrm{E}_{g}\{(\overline{z^{2}}-1)^{2}\overline{z}^{4}\}\\
&=6[\ n\,\mathrm{var}_{g}(\overline{z^{2}})3\{n\,\mathrm{var}_{g}(\overline{z})\}^{2}+12\{n\,\mathrm{cov}_{g}(\overline{z^{2}},\overline{z})\}^{2}n\,\mathrm{var}_{g}(\overline{z})\ ]+O(n^{-1})\\
&=6\{(\kappa_{4}+2)\times 3+12\kappa_{3}^{2}\}+O(n^{-1})=18(\kappa_{4}+4\kappa_{3}^{2}+2)+O(n^{-1}).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\kappa_{g4}(n^{-1}\mathrm{AIC}_{\mathrm{ML}})&=n^{-3}[\ \kappa_{8}+24\kappa_{6}+56\kappa_{5}\kappa_{3}+32\kappa_{4}^{2}+144\kappa_{4}+240\kappa_{3}^{2}+48\\
&\quad-4(\kappa_{6}+24\kappa_{4}+58\kappa_{3}^{2}+3\kappa_{4}^{2}+6\kappa_{3}\kappa_{5}+20)+18(\kappa_{4}+4\kappa_{3}^{2}+2)\\
&\quad-4\times(-1)(\kappa_{6}+12\kappa_{4}+4\kappa_{3}^{2}+8)-6\times(-2)(\kappa_{4}+2)(\kappa_{4}+1)\\
&\quad-6(\kappa_{4}+2)(-1)^{2}\ \ ]+O(n^{-4})\\
&=n^{-3}[\ \kappa_{8}+(24-4+4)\kappa_{6}+(56-24)\kappa_{5}\kappa_{3}+(32-12+12)\kappa_{4}^{2}\\
&\quad+(144-96+18+48+36-6)\kappa_{4}+(240-4\times 58+18\times 4+4\times 4)\kappa_{3}^{2}\\
&\quad+48-80+36+32+24-12\ \ ]+O(n^{-4})\\
&=n^{-3}(\kappa_{8}+24\kappa_{6}+32\kappa_{5}\kappa_{3}+32\kappa_{4}^{2}+144\kappa_{4}+96\kappa_{3}^{2}+48)+O(n^{-4})\\
&=n^{-3}\alpha_{\mathrm{ML4}}^{(\mathrm{A})}+O(n^{-4})
\end{aligned}
\qquad \text{(S6.11)}
$$

(under normality $\alpha_{\mathrm{ML4}}^{(\mathrm{A})}=48$).

S6.2.2 Asymptotic cumulants of $n^{-1}\mathrm{AIC}_{\mathrm{ML}}$ after studentization for estimation of $-2\overline{l}_{0}^{*}$

$$
t_{\mathrm{ML}}^{(\mathrm{A})}=\frac{n^{1/2}(n^{-1}\mathrm{AIC}_{\mathrm{ML}}+2\overline{l}_{0}^{*})}{(\hat{v}_{\mathrm{ML}}^{(\mathrm{A})})^{1/2}}.
$$

$$
\begin{aligned}
\kappa_{g1}(t_{\mathrm{ML}}^{(\mathrm{A})})&=n^{-1/2}\{\alpha_{\mathrm{ML1}}^{(\mathrm{A})}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}+\alpha_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})}\}+O(n^{-3/2})\\
&=n^{-1/2}\{\alpha_{\mathrm{ML1}}^{(\mathrm{A})}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}+n\mathrm{E}_{g}(\overline{l}_{\mathrm{ML}}^{(1)}\mathbf{m}_{v}^{(1)}{}')\mathbf{v}^{(1)}\}+O(n^{-3/2})\\
&=n^{-1/2}\left\{(\kappa_{4}+2)^{-1/2}-\frac{1}{2}(\kappa_{4}+2)^{-3/2}(\kappa_{6}+12\kappa_{4}+6\kappa_{3}^{2}+8)\right\}\\
&\quad+O(n^{-3/2})\\
&=n^{-1/2}\alpha_{(t)\mathrm{ML1}}^{(\mathrm{A})}+O(n^{-3/2})
\end{aligned}
\qquad \text{(S6.12)}
$$

($\alpha^{(\mathrm{A})}_{(\Delta t)\mathrm{ML1}} = -(1/2)(\kappa_4+2)^{-3/2}(\kappa_6+12\kappa_4+6\kappa_3^2+8)$; under normality $\alpha^{(\mathrm{A})}_{(t)\mathrm{ML1}} = 2^{-1/2}-2^{1/2}$ and $\alpha^{(\mathrm{A})}_{(\Delta t)\mathrm{ML1}} = -2^{1/2}$),

$$
\begin{aligned}
&\kappa_{g2}(t^{(\mathrm{A})}_{\mathrm{ML}})\\
&= 1 + n^{-1}\underset{(\mathrm{A})}{\Bigg[}\ \alpha^{(\mathrm{A})}_{\mathrm{ML}\Delta 2}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1} + 2n^2\mathrm{E}_g\{(\bar{l}^{(1)}_{\mathrm{ML}})^2\mathbf{v}^{(1)}{}'\mathbf{m}^{(1)}_v\}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}\\
&\quad + 2n^2\mathrm{E}_g\underset{(\mathrm{B})}{\Bigg[}\ \bar{l}^{(1)}_{\mathrm{ML}}(\bar{l}^{(2)}_{\mathrm{ML}}\mathbf{v}^{(1)}{}'\mathbf{m}^{(1)}_v + \bar{l}^{(1)}_{\mathrm{ML}}\mathbf{v}^{(2)}{}'\mathbf{m}^{(2)}_v + n^{-1}2q\mathbf{v}^{(1)}{}'\mathbf{m}^{(1)}_v)\ \underset{(\mathrm{B})}{\Bigg]}\\
&\qquad\qquad \times(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2} \qquad\qquad\qquad\qquad\qquad\qquad (\mathrm{S6.13})\\
&\qquad + n^2\mathrm{E}_g\{2\bar{l}^{(2)}_{\mathrm{ML}}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}\bar{l}^{(1)}_{\mathrm{ML}}\mathbf{v}^{(1)}{}'\mathbf{m}^{(1)}_v + (\bar{l}^{(1)}_{\mathrm{ML}}\mathbf{v}^{(1)}{}'\mathbf{m}^{(1)}_v)^2\}\\
&\qquad - \{2n\mathrm{E}_g(\bar{l}^{(2)}_{\mathrm{ML}})(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}\alpha^{(\mathrm{A})}_{(\Delta t)\mathrm{ML1}} + (\alpha^{(\mathrm{A})}_{(\Delta t)\mathrm{ML1}})^2\}\ \underset{(\mathrm{A})}{\Bigg]} + O(n^{-2}).
\end{aligned}
$$

(i) the first term in $\underset{(\mathrm{A})}{[}\,\cdot\,\underset{(\mathrm{A})}{]}$ of (S6.13) is

$\alpha^{(\mathrm{A})}_{\mathrm{ML}\Delta 2}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1} = -2(\kappa_4+1)(\kappa_4+2)^{-1}$ ($=-1$ under normality),

(ii) the second term in $\underset{(\mathrm{A})}{[}\,\cdot\,\underset{(\mathrm{A})}{]}$ of (S6.13) is

$$
\begin{aligned}
&2n^2\mathrm{E}_g\{(\bar{l}^{(1)}_{\mathrm{ML}})^2\mathbf{v}^{(1)}{}'\mathbf{m}^{(1)}_v\}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}\\
&= 2\left\{16\kappa_{g4}(l_{0j}),\ \frac{1}{\sigma}(\kappa_5+8\kappa_3)\right\}\frac{1}{2}(\kappa_4+2)^{-3/2}(-1,\ 4\sigma\kappa_3)'(\kappa_4+2)^{-1/2}\\
&= (\kappa_4+2)^{-2}\{-16\kappa_{g4}(l_{0j}) + 4\kappa_3(\kappa_5+8\kappa_3)\}\\
&= (\kappa_4+2)^{-2}\{-(\kappa_8+24\kappa_6+56\kappa_5\kappa_3+32\kappa_4^2+144\kappa_4+240\kappa_3^2+48)\\
&\qquad\qquad + 4\kappa_3(\kappa_5+8\kappa_3)\}\\
&= -(\kappa_4+2)^{-2}(\kappa_8+24\kappa_6+52\kappa_5\kappa_3+32\kappa_4^2+144\kappa_4+208\kappa_3^2+48)
\end{aligned}
$$

($=-12$ under normality),

(iii) the first part in $2n^{-2}\mathrm{E}_g \underset{\text{(B)}}{[}\,\cdot\,\underset{\text{(B)}}{]}\,(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}$ of the third term in $\underset{\text{(A)}}{[}\,\cdot\,\underset{\text{(A)}}{]}$ of (S6.13) is

$$
\begin{aligned}
&2n^2\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(1)}\bar{l}_{\mathrm{ML}}^{(2)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)})(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\\
&=2\Bigg[\ n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(2)})n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)}\\
&\quad+2n\mathrm{E}_g\left(\bar{l}_{\mathrm{ML}}^{(1)}\frac{\partial\bar{l}}{\partial\theta_0}\right)\lambda^{-1}\left\{n\mathrm{E}_g\left(\frac{\partial\bar{l}}{\partial\theta_0}m_v\right),\gamma\right\}\mathbf{v}^{(1)}\Bigg](\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}+O(n^{-1})\\
&=2\Bigg[(-1)\left\{-\frac{1}{2}(\kappa_4+2)^{-3/2}(\kappa_6+12\kappa_4+6\kappa_3^2+8)\right\}\\
&\quad+2\frac{\kappa_3}{\sigma}(-\sigma^2)\left\{n\mathrm{E}_g\left(\frac{\partial\bar{l}}{\partial\theta_0}m_v\right),\gamma\right\}\frac{1}{2}(\kappa_4+2)^{-3/2}(-1,\,4\sigma\kappa_3)'\Bigg](\kappa_4+2)^{-1/2}\\
&\quad+O(n^{-1})\\
&=(\kappa_4+2)^{-2}\left[\kappa_6+12\kappa_4+6\kappa_3^2+8-2\kappa_3\sigma\left\{\frac{1}{\sigma}(\kappa_5+8\kappa_3),\,\frac{1}{\sigma^2}\right\}(-1,\,4\sigma\kappa_3)'\right]\\
&\quad+O(n^{-1})\\
&=(\kappa_4+2)^{-2}[\ \kappa_6+12\kappa_4+6\kappa_3^2+8-2\kappa_3\{-(\kappa_5+8\kappa_3)+4\kappa_3\}\ ]+O(n^{-1})\\
&=(\kappa_4+2)^{-2}(\kappa_6+12\kappa_4+14\kappa_3^2+2\kappa_3\kappa_5+8)+O(n^{-1})
\end{aligned}
$$

(= 2 under normality),

(iv) the central part in $2n^{-2}\mathrm{E}_g \underset{\text{(B)}}{[}\,\cdot\,\underset{\text{(B)}}{]}\,(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}$ of the third term in $\underset{\text{(A)}}{[}\,\cdot\,\underset{\text{(A)}}{]}$ of (S6.13) is

$$
\begin{aligned}
&2n^2\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^2\mathbf{v}^{(2)}{}'\mathbf{m}_v^{(2)}\}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\\
&=2\underset{\text{(C)}}{\Bigg[}\ \alpha_{\mathrm{ML2}}^{(\mathrm{A})}\left[n\,\mathrm{avar}_g(m_v),n\,\mathrm{cov}_g\left(m_v,\frac{\partial\bar{l}}{\partial\theta_0}\right),0,\gamma,n\,\mathrm{cov}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial\theta_0},\frac{\partial\bar{l}}{\partial\theta_0}\right)\right]\mathbf{v}^{(2)}
\end{aligned}
$$

$$+8\Bigg[_{(\mathrm{D})}\ \{n\,\mathrm{cov}_g(\bar{l}_0,m_v)\}^2,\ n\,\mathrm{cov}_g(\bar{l}_0,m_v)\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial\theta_0}\right),0,$$

$$\left\{\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial\theta_0}\right)\right\}^2,n\,\mathrm{cov}_g\left(\bar{l}_0,\frac{\partial v^{(\mathrm{A})}}{\partial\theta_0}\right)\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial\theta_0}\right)\ \Bigg]_{(\mathrm{D})}\mathbf{v}^{(2)}\ \Bigg]_{(\mathrm{C})}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}$$

$$+O(n^{-1})$$

$$=2\Bigg[_{(\mathrm{C})}\ [\,(\kappa_4+2)n\,\mathrm{avar}_g(m_v)+8\{n\,\mathrm{cov}_g(\bar{l}_0,m_v)\}^2\ ]\frac{3}{8}(\kappa_4+2)^{-3}$$

$$+\left\{(\kappa_4+2)n\,\mathrm{cov}_g\left(m_v,\frac{\partial\bar{l}}{\partial\theta_0}\right)+8n\,\mathrm{cov}_g(\bar{l}_0,m_v)\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial\theta_0}\right)\right\}$$

$$\times\{-3(\kappa_4+2)^{-3}\kappa_3\sigma\}$$

$$+\left[(\kappa_4+2)\gamma+8\left\{\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial\theta_0}\right)\right\}^2\right]\{-2(\kappa_4+2)^{-2}+6(\kappa_4+2)^{-3}\kappa_3^2\}\sigma^2$$

$$+\left\{(\kappa_4+2)n\,\mathrm{cov}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial\theta_0},\frac{\partial\bar{l}}{\partial\theta_0}\right)+8n\,\mathrm{cov}_g\left(\bar{l}_0,\frac{\partial v^{(\mathrm{A})}}{\partial\theta_0}\right)\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial\theta_0}\right)\right\}$$

$$\times\left\{-\frac{\sigma^2}{2}(\kappa_4+2)^{-2}\right\}\ \Bigg]_{(\mathrm{C})}+O(n^{-1})$$

$$=2\Big[_{(\mathrm{C})}\ \{\,(\kappa_4+2)(\kappa_8+24\kappa_6+56\kappa_5\kappa_3+34\kappa_4^2+152\kappa_4+240\kappa_3^2+56)$$

$$+2(\kappa_6+12\kappa_4+10\kappa_3^2+8)^2\ \}\frac{3}{8}(\kappa_4+2)^{-3}$$

$$+\left\{(\kappa_4+2)\frac{1}{\sigma}(\kappa_5+8\kappa_3)+2(\kappa_6+12\kappa_4+10\kappa_3^2+8)\frac{\kappa_3}{\sigma}\right\}\{-3(\kappa_4+2)^{-3}\kappa_3\sigma\}$$

$$+\left\{(\kappa_4+2)\frac{1}{\sigma^2}+2\frac{\kappa_3^2}{\sigma^2}\right\}\{-2(\kappa_4+2)^{-2}+6(\kappa_4+2)^{-3}\kappa_3^2\}\sigma^2$$

$$+\left\{-(\kappa_4+2)\frac{4}{\sigma^2}(\kappa_4+2)-\frac{8}{\sigma}(\kappa_5+8\kappa_3)\frac{\kappa_3}{\sigma}\right\}\left\{-\frac{\sigma^2}{2}(\kappa_4+2)^{-2}\right\}\ \Big]_{(\mathrm{C})}+O(n^{-1})$$

$$=2\Big[_{(\mathrm{C})}\ (\kappa_4+2)^{-3}\left\{\frac{3}{4}(\kappa_6+12\kappa_4+10\kappa_3^2+8)^2\right.$$

$$\left.-6(\kappa_6+12\kappa_4+10\kappa_3^2+8)\kappa_3^2+12\kappa_3^4\ \right\}$$

$$+(\kappa_4+2)^{-2}\left\{\frac{3}{8}(\kappa_8+24\kappa_6+56\kappa_5\kappa_3+34\kappa_4^2+152\kappa_4+240\kappa_3^2+56)\right.$$

$$\left.-3(\kappa_5+8\kappa_3)\kappa_3+6\kappa_3^2-4\kappa_3^2+4(\kappa_5+8\kappa_3)\kappa_3\ \right\}$$

$$+(\kappa_4+2)^{-1}(-2)+2\ \Big]_{(\mathrm{C})}+O(n^{-1})$$

$$=2\underset{\text{(C)}}{\Bigg[}\ (\kappa_4+2)^{-3}\frac{3}{4}(\kappa_6+12\kappa_4+6\kappa_3^2+8)^2$$

$$+(\kappa_4+2)^{-2}\left\{\frac{3}{8}(\kappa_8+24\kappa_6+34\kappa_4^2+152\kappa_4+56)+22\kappa_5\kappa_3+100\kappa_3^2\right\}$$

$$-2(\kappa_4+2)^{-1}+2\ \underset{\text{(C)}}{\Bigg]}+O(n^{-1})$$

($=2(2^{-3}\times 48+2^{-2}\times 3\times 7-2\times 2^{-1}+2)=2\{6+(21/4)+1\}=49/2$ under normality),

(v) the third part in $2n^{-2}\mathrm{E}_g\underset{\text{(B)}}{[}\ \cdot\ \underset{\text{(B)}}{]}(\alpha^{(\mathrm{A})}_{\mathrm{ML}2})^{-1/2}$ of the third term in $\underset{\text{(A)}}{[}\ \cdot\ \underset{\text{(A)}}{]}$ of (S6.13) is

$$2n^2\mathrm{E}_g\{n^{-1}\bar{l}^{(1)}_{\mathrm{ML}}2q\mathbf{m}^{(1)}_v{}'\mathbf{v}^{(1)}\}(\alpha^{(\mathrm{A})}_{\mathrm{ML}2})^{-1/2}=4n\mathrm{E}_g(\bar{l}^{(1)}_{\mathrm{ML}}\mathbf{m}^{(1)}_v{}')\mathbf{v}^{(1)}(\alpha^{(\mathrm{A})}_{\mathrm{ML}2})^{-1/2}$$

$$=4\left\{-\frac{1}{2}(\kappa_4+2)^{-3/2}(\kappa_6+12\kappa_4+6\kappa_3^2+8)\right\}(\kappa_4+2)^{-1/2}$$

$$=-2(\kappa_4+2)^{-2}(\kappa_6+12\kappa_4+6\kappa_3^2+8)$$

($=-2\times 2^{-2}\times 8=-4$ under normality),

(vi) the first half of the fourth term in $\underset{\text{(A)}}{[}\ \cdot\ \underset{\text{(A)}}{]}$ of (S6.13)

$2n^2\mathrm{E}_g\{\bar{l}^{(2)}_{\mathrm{ML}}(\alpha^{(\mathrm{A})}_{\mathrm{ML}2})^{-1/2}\bar{l}^{(1)}_{\mathrm{ML}}\mathbf{v}^{(1)}{}'\mathbf{m}^{(1)}_v\}$ is equal to the value in (iii),

(vii) the second half of the fourth term in $\underset{\text{(A)}}{[}\ \cdot\ \underset{\text{(A)}}{]}$ of (S6.13) is

$$n^2\mathrm{E}_g\{(\bar{l}^{(1)}_{\mathrm{ML}}\mathbf{v}^{(1)}{}'\mathbf{m}^{(1)}_v)^2\}$$

$$=\alpha^{(\mathrm{A})}_{\mathrm{ML}2}\mathbf{v}^{(1)}{}'n\,\mathrm{acov}_g(\mathbf{m}^{(1)}_v)\mathbf{v}^{(1)}+2\{n\mathrm{E}_g(\bar{l}^{(1)}_{\mathrm{ML}}\mathbf{m}^{(1)}_v{}')\mathbf{v}^{(1)}\}^2+O(n^{-1})$$

$$
=(\kappa_4+2)\frac{1}{4}(\kappa_4+2)^{-3}(-1,\ 4\sigma\kappa_3)
$$

$$
\times\begin{bmatrix} n\,\mathrm{avar}_g(m_v) & n\,\mathrm{cov}_g(m_v,\partial\bar{l}\,/\,\partial\theta_0) \\ n\,\mathrm{cov}_g(m_v,\partial\bar{l}\,/\,\partial\theta_0) & \gamma \end{bmatrix}\begin{pmatrix}-1\\ 4\sigma\kappa_3\end{pmatrix}
$$

$$
+2\times\frac{1}{4}(\kappa_4+2)^{-3}(\kappa_6+12\kappa_4+6\kappa_3^2+8)^2+O(n^{-1})
$$

$$
=\frac{1}{4}(\kappa_4+2)^{-2}(-1,\ 4\sigma\kappa_3)
$$

$$
\times\begin{bmatrix} \begin{array}{l}\kappa_8+24\kappa_6+56\kappa_5\kappa_3+34\kappa_4^2\\ +152\kappa_4+240\kappa_3^2+56\end{array} & \dfrac{1}{\sigma}(\kappa_5+8\kappa_3) \\ \dfrac{1}{\sigma}(\kappa_5+8\kappa_3) & \dfrac{1}{\sigma^2} \end{bmatrix}\begin{pmatrix}-1\\ 4\sigma\kappa_3\end{pmatrix}
$$

$$
+\frac{1}{2}(\kappa_4+2)^{-3}(\kappa_6+12\kappa_4+6\kappa_3^2+8)^2+O(n^{-1})
$$

$$
=\frac{1}{4}(\kappa_4+2)^{-2}\{\kappa_8+24\kappa_6+(56-8)\kappa_5\kappa_3+34\kappa_4^2+152\kappa_4
$$

$$
+(240-64+16)\kappa_3^2+56\}
$$

$$
+\frac{1}{2}(\kappa_4+2)^{-3}(\kappa_6+12\kappa_4+6\kappa_3^2+8)^2+O(n^{-1})
$$

$$
=\frac{1}{4}(\kappa_4+2)^{-2}(\kappa_8+24\kappa_6+48\kappa_5\kappa_3+34\kappa_4^2+152\kappa_4+192\kappa_3^2+56)
$$

$$
+\frac{1}{2}(\kappa_4+2)^{-3}(\kappa_6+12\kappa_4+6\kappa_3^2+8)^2+O(n^{-1})
$$

$$
\left(=\frac{1}{4}\times\frac{1}{4}\times 56+\frac{1}{2}\times\frac{1}{8}\times 64=\frac{7}{2}+4=\frac{15}{2}\ \text{under normality}\right),
$$

(viii) the fifth term in $\underset{(\mathrm{A})}{[}\ \cdot\ \underset{(\mathrm{A})}{]}$ of (S6.13) is

$$-\{2n\mathrm{E}_g(\bar{l}^{(2)}_{\mathrm{ML}})(\alpha^{(\mathrm{A})}_{\mathrm{ML}2})^{-1/2}\alpha^{(\mathrm{A})}_{(\Delta t)\mathrm{ML}1}+(\alpha^{(\mathrm{A})}_{(\Delta t)\mathrm{ML}1})^2\}$$

$$=-\Bigg[2(-1)(\kappa_4+2)^{-1/2}\left\{-\frac{1}{2}(\kappa_4+2)^{-3/2}(\kappa_6+12\kappa_4+6\kappa_3^2+8)\right\}$$

$$+\frac{1}{4}(\kappa_4+2)^{-3}(\kappa_6+12\kappa_4+6\kappa_3^2+8)^2\Bigg]$$

$$=-\left\{(\kappa_4+2)^{-2}(\kappa_6+12\kappa_4+6\kappa_3^2+8)+\frac{1}{4}(\kappa_4+2)^{-3}(\kappa_6+12\kappa_4+6\kappa_3^2+8)^2\right\}$$

$(=-(2^{-2}\times 8+(1/4)2^{-3}\times 8^2)=-4$ under normality).

Consequently,

$$\kappa_{g2}(t^{(\mathrm{A})}_{\mathrm{ML}})=1+n^{-1}\Big[_{(\mathrm{A})}\ -2(\kappa_4+2)^{-1}(\kappa_4+1)$$

$$-(\kappa_4+2)^{-2}(\kappa_8+24\kappa_6+52\kappa_5\kappa_3+32\kappa_4^2+144\kappa_4+208\kappa_3^2+48)$$

$$+2(\kappa_4+2)^{-2}(\kappa_6+12\kappa_4+14\kappa_3^2+2\kappa_5\kappa_3+8)$$

$$+2\Big[_{(\mathrm{B})}\ (\kappa_4+2)^{-3}\frac{3}{4}(\kappa_6+12\kappa_4+6\kappa_3^2+8)^2$$

$$+(\kappa_4+2)^{-2}\left\{\frac{3}{8}(\kappa_8+24\kappa_6+34\kappa_4^2+152\kappa_4+56)+22\kappa_5\kappa_3+100\kappa_3^2\right\}$$

$$-2(\kappa_4+2)^{-1}+2\ \Big]_{(\mathrm{B})} \tag{S6.14}$$

$$-2(\kappa_4+2)^{-2}(\kappa_6+12\kappa_4+6\kappa_3^2+8)$$

$$+\frac{1}{4}(\kappa_4+2)^{-2}(\kappa_8+24\kappa_6+48\kappa_5\kappa_3+34\kappa_4^2+152\kappa_4+192\kappa_3^2+56)$$

$$+\frac{1}{2}(\kappa_4+2)^{-3}(\kappa_6+12\kappa_4+6\kappa_3^2+8)^2$$

$$-\left\{(\kappa_4+2)^{-2}(\kappa_6+12\kappa_4+6\kappa_3^2+8)+\frac{1}{4}(\kappa_4+2)^{-3}(\kappa_6+12\kappa_4+6\kappa_3^2+8)^2\right\}\Bigg]_{\text{(A)}}$$

$$+O(n^{-2})$$

$$=1+n^{-1}\Bigg[_{\text{(A)}}\ 4-2(\kappa_4+2)^{-1}(\kappa_4+1+2)$$

$$+(\kappa_4+2)^{-2}\{\ (-1+(3/4)+(1/4))\kappa_8+(-24+2+18-2+6-1)\kappa_6$$

$$+(-52+4+44+12)\kappa_5\kappa_3+(-32+(3/2)17+(17/2))\kappa_4^2$$

$$+(-144+24+6\times 19-24+38-12)\kappa_4$$

$$+(-208+28+200-12+48-6)\kappa_3^2-48+16+6\times 7-16+14-8\ \}$$

$$+(\kappa_4+2)^{-3}\left(\frac{3}{2}+\frac{1}{2}-\frac{1}{4}\right)(\kappa_6+12\kappa_4+6\kappa_3^2+8)^2\ \Bigg]_{\text{(A)}}+O(n^{-2})$$

$$=1+n^{-1}\Bigg\{\ 2-2(\kappa_4+2)^{-1}+(\kappa_4+2)^{-2}(-\kappa_6+8\kappa_5\kappa_3+2\kappa_4^2-4\kappa_4+50\kappa_3^2)$$

$$+(\kappa_4+2)^{-3}\frac{7}{4}(\kappa_6+12\kappa_4+6\kappa_3^2+8)^2\Bigg\}+O(n^{-2})$$

$$=1+n^{-1}\alpha^{\text{(A)}}_{(t)\mathrm{ML}\Delta 2}+O(n^{-2})\quad(\alpha^{\text{(A)}}_{(t)\mathrm{ML}2}=1)$$

$(=1+n^{-1}\{2-1+(7/4)8\}+O(n^{-2})=1+n^{-1}15+O(n^{-2})$ under normality),

$$
\begin{aligned}
\kappa_{g3}(t^{(\mathrm{A})}_{\mathrm{ML}}) &= n^{-1/2}\{\alpha^{(\mathrm{A})}_{\mathrm{ML3}}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-3/2} + 6\alpha^{(\mathrm{A})}_{(\Delta t)\mathrm{ML1}}\} + O(n^{-3/2}) \\
&= \left\{ \begin{aligned} &(\kappa_6 + 12\kappa_4 + 4\kappa_3^2 + 8)(\kappa_4 + 2)^{-3/2} \\ &-6\times\frac{1}{2}(\kappa_4+2)^{-3/2}(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8) \end{aligned} \right\} + O(n^{-3/2}) \\
&= -n^{-1/2}2(\kappa_4+2)^{-3/2}(\kappa_6 + 12\kappa_4 + 7\kappa_3^2 + 8) + O(n^{-3/2}) \\
&= n^{-1/2}\alpha^{(\mathrm{A})}_{(t)\mathrm{ML3}} + O(n^{-3/2})
\end{aligned}
\tag{S6.15}
$$

(under normality $\alpha^{(\mathrm{A})}_{(t)\mathrm{ML3}} = -2\times 2^{3/2} = -2^{5/2}$).

$$
\begin{aligned}
&\kappa_{g4}(t^{(\mathrm{A})}_{\mathrm{ML}}) \\
&= n^{-1}\Bigg[_{(\mathrm{A})} \quad \alpha^{(\mathrm{A})}_{\mathrm{ML4}}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-2} + 4n^3\mathrm{E}_g\{(\bar{l}^{(1)}_{\mathrm{ML}})^4\mathbf{v}^{(1)\prime}\mathbf{m}^{(1)}_v\}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-3/2} \\
&\quad + 12n^3\mathrm{E}_g\{(\bar{l}^{(1)}_{\mathrm{ML}})^3\bar{l}^{(2)}_{\mathrm{ML}}\mathbf{v}^{(1)\prime}\mathbf{m}^{(1)}_v\}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-3/2} \\
&\quad + 6n^3\mathrm{E}_g\{(\bar{l}^{(1)}_{\mathrm{ML}})^4(\mathbf{v}^{(1)\prime}\mathbf{m}^{(1)}_v)^2\}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1} \\
&\quad + 4n^3\mathrm{E}_g\{(\bar{l}^{(1)}_{\mathrm{ML}})^3\bar{l}^{(2)}_{\mathrm{ML}}\mathbf{v}^{(1)\prime}\mathbf{m}^{(1)}_v + (\bar{l}^{(1)}_{\mathrm{ML}})^4\mathbf{v}^{(2)\prime}\mathbf{m}^{(2)}_v\}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-3/2} \\
&\quad + [4n\mathrm{E}_g(\bar{l}^{(2)}_{\mathrm{ML}})\alpha^{(\mathrm{A})}_{\mathrm{ML3}} + 6\alpha^{(\mathrm{A})}_{\mathrm{ML2}}\alpha^{(\mathrm{A})}_{\mathrm{ML\Delta 2}} + 6\alpha^{(\mathrm{A})}_{\mathrm{ML2}}\{n\mathrm{E}_g(\bar{l}^{(2)}_{\mathrm{ML}})\}^2](\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-2} \\
&\quad - 4\{\alpha^{(\mathrm{A})}_{(t)\mathrm{ML1}} - 2q(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}\}\alpha^{(\mathrm{A})}_{(t)\mathrm{ML3}} \\
&\quad - 6\{\alpha^{(\mathrm{A})}_{(t)\mathrm{ML\Delta 2}} - 4qn\mathrm{E}_g(\bar{l}^{(1)}_{\mathrm{ML}}\mathbf{m}^{(1)\prime}_v)\mathbf{v}^{(1)}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}\} \\
&\quad - 6\{\alpha^{(\mathrm{A})}_{(t)\mathrm{ML1}} - 2q(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}\}^2 \quad \Bigg]_{(\mathrm{A})} + O(n^{-2}),
\end{aligned}
\tag{S6.16}
$$

(i) the first term in $[_{(\mathrm{A})}\ \cdot\ ]_{(\mathrm{A})}$ of (S6.16) is

$$
\begin{aligned}
\alpha^{(\mathrm{A})}_{\mathrm{ML4}}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-2} &= (\kappa_8 + 24\kappa_6 + 32\kappa_5\kappa_3 + 32\kappa_4^2 + 144\kappa_4 + 96\kappa_3^2 + 48) \\
&\quad \times(\kappa_4+2)^{-2}
\end{aligned}
$$

(= 12 under normality),

(ii) the second term in $\underset{(\mathrm{A})}{[}\cdot\underset{(\mathrm{A})}{]}$ of (S6.16) is

$$4n^3\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^4\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}\}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}$$

$$=4\underset{(\mathrm{B})}{\Bigg[}\ 6\times 4\alpha_{\mathrm{ML2}}^{(\mathrm{A})}\left[4\kappa_{g4}(l_{0j}),\ \mathrm{E}_g\left\{\frac{\partial l_j}{\partial\theta_0}(l_{0j}-\bar{l}_0^*)^2\right\}\right]$$

$$-4\times 8\kappa_{g3}(l_{0j})\left\{n\,\mathrm{cov}_g(\bar{l}_{\mathrm{ML}}^{(1)},m_v),-2\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial\theta_0}\right)\right\}\ \underset{(\mathrm{B})}{\Bigg]}\mathbf{v}^{(1)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}+O(n^{-1})$$

$$=\underset{(\mathrm{B})}{\Bigg[}\ 96(\kappa_4+2)$$

$$\times\left\{\frac{1}{4}(\kappa_8+24\kappa_6+56\kappa_5\kappa_3+32\kappa_4^2+144\kappa_4+240\kappa_3^2+48),\ \frac{1}{4\sigma}(\kappa_5+8\kappa_3)\right\}$$

$$-128\left\{-\frac{1}{8}(\kappa_6+12\kappa_4+10\kappa_3^2+8)\right\}\left(\kappa_6+12\kappa_4+10\kappa_3^2+8,\ \frac{\kappa_3}{\sigma}\right)\ \underset{(\mathrm{B})}{\Bigg]}$$

$$\times\frac{1}{2}(\kappa_4+2)^{-3/2}(-1,\ 4\sigma\kappa_3)'(\kappa_4+2)^{-3/2}+O(n^{-1})$$

$$=12(\kappa_4+2)^{-2}\{-(\kappa_8+24\kappa_6+56\kappa_5\kappa_3+32\kappa_4^2+144\kappa_4+240\kappa_3^2+48)$$
$$+4(\kappa_5+8\kappa_3)\kappa_3\ \}$$
$$+8(\kappa_4+2)^{-3}\{-(\kappa_6+12\kappa_4+10\kappa_3^2+8)^2+4(\kappa_6+12\kappa_4+10\kappa_3^2+8)\kappa_3^2\ \}$$
$$+O(n^{-1})$$

$$=-12(\kappa_4+2)^{-2}(\kappa_8+24\kappa_6+52\kappa_5\kappa_3+32\kappa_4^2+144\kappa_4+208\kappa_3^2+48)$$
$$-8(\kappa_4+2)^{-3}\{(\kappa_6+12\kappa_4+8\kappa_3^2+8)^2-4\kappa_3^4\ \}$$

$(=-12(1/4)\times 48-8(1/8)\times 64=-144-64=-208$ under normality),

(iii) the third term in $\underset{(\mathrm{A})}{[}\cdot\underset{(\mathrm{A})}{]}$ of (S6.16) is

$$\begin{aligned}
&12n^3\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^3\bar{l}_{\mathrm{ML}}^{(2)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}\}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\\
&=12\Bigg[\ 3\alpha_{\mathrm{ML2}}^{(\mathrm{A})}n^2\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(1)}\bar{l}_{\mathrm{ML}}^{(2)}\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\\
&\qquad+6\left\{n\,\mathrm{cov}_g\left(\bar{l}_{\mathrm{ML}}^{(1)},\frac{\partial\bar{l}}{\partial\theta_0}\right)\right\}^2\lambda^{-1}n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(1)}\,\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\ \Bigg]+O(n^{-1})\\
&=36n^2\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(1)}\bar{l}_{\mathrm{ML}}^{(2)}\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\\
&\quad+12\times6\times(-2)^2\left\{\mathrm{E}_g\left(l_{0j},\frac{\partial l_j}{\partial\theta_0}\right)\right\}^2(-\sigma^2)n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(1)}\,\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\\
&\quad+O(n^{-1})\\
&=18(\kappa_4+2)^{-2}(\kappa_6+12\kappa_4+14\kappa_3^2+2\kappa_5\kappa_3+8)\\
&+72\frac{\kappa_3^2}{\sigma^2}(-\sigma^2)\left\{-\frac{1}{2}(\kappa_4+2)^{-3/2}(\kappa_6+12\kappa_4+6\kappa_3^2+8)\right\}(\kappa_4+2)^{-3/2}+O(n^{-1})\\
&=18(\kappa_4+2)^{-2}(\kappa_6+12\kappa_4+14\kappa_3^2+2\kappa_5\kappa_3+8)\\
&\quad+36(\kappa_4+2)^{-3}\kappa_3^2(\kappa_6+12\kappa_4+6\kappa_3^2+8)+O(n^{-1})
\end{aligned}$$

($=18(1/4)\times8=36$ under normality; note that the first term $18(\kappa_4+2)^{-2}(\cdot)$ on the left-hand side of the above last equation is 18 times the result of (iii) for (S6.13) of $\alpha_{(t)\mathrm{ML}\Delta2}^{(\mathrm{A})}$)

(iv) the fourth term in $\underset{(\mathrm{A})}{[}\,\cdot\,\underset{(\mathrm{A})}{]}$ of (S6.16) is

$$\begin{aligned}
&6n^3\mathrm{E}_g\{(\bar{l}_{\mathrm{ML}}^{(1)})^4(\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)})^2\}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1}\\
&=6[\ 3(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^2\mathbf{v}^{(1)}{}'n\,\mathrm{acov}_g(\mathbf{m}_v^{(1)})\mathbf{v}^{(1)}\\
&\qquad+12\alpha_{\mathrm{ML2}}^{(\mathrm{A})}\{n\,\mathrm{cov}_g(\bar{l}_{\mathrm{ML}}^{(1)},\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)}\}^2\ ](\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1}+O(n^{-1})\\
&=18\times\frac{1}{4}(\kappa_4+2)^{-2}(\kappa_8+24\kappa_6+48\kappa_5\kappa_3+34\kappa_4^2+152\kappa_4+192\kappa_3^2+56)\\
&\quad+36\times\frac{1}{2}(\kappa_4+2)^{-3}(\kappa_6+12\kappa_4+6\kappa_3^2+8)^2+O(n^{-1})
\end{aligned}$$

$$=\frac{9}{2}(\kappa_4+2)^{-2}(\kappa_8+24\kappa_6+48\kappa_5\kappa_3+34\kappa_4^2+152\kappa_4+192\kappa_3^2+56)$$

$$+18(\kappa_4+2)^{-3}(\kappa_6+12\kappa_4+6\kappa_3^2+8)^2+O(n^{-1})$$

$\Bigg($ $=\frac{9}{2}\times\frac{1}{4}\times 56+18\times\frac{1}{8}\times 64=63+144=207$ under normality; see the result of (vii) for (S6.13) of $\alpha^{(\mathrm{A})}_{(t)\mathrm{ML}\Delta 2}$ $\Bigg)$,

(v) the first half of the fifth term in $\underset{(\mathrm{A})}{[}\ \cdot\ \underset{(\mathrm{A})}{]}$ of (S6.16) is

$$4n^3\mathrm{E}_g\{(\bar{l}^{(1)}_{\mathrm{ML}})^3\bar{l}^{(2)}_{\mathrm{ML}}\mathbf{v}^{(1)}{}'\mathbf{m}^{(1)}_v\}(\alpha^{(\mathrm{A})}_{\mathrm{ML}2})^{-3/2}$$

(= 36/3 =12 under normality; one third of the value in (iii))

(vi) the second half of the fifth term in $\underset{(\mathrm{A})}{[}\ \cdot\ \underset{(\mathrm{A})}{]}$ of (S6.16) is

$$4n^3\mathrm{E}_g\{(\bar{l}^{(1)}_{\mathrm{ML}})^4\mathbf{v}^{(2)}{}'\mathbf{m}^{(2)}_v\}(\alpha^{(\mathrm{A})}_{\mathrm{ML}2})^{-3/2}$$

$$=4\underset{(\mathrm{B})}{\Bigg[}\ 3(\alpha^{(\mathrm{A})}_{\mathrm{ML}2})^2\left\{n\,\mathrm{avar}_g(m_v),n\,\mathrm{cov}_g\left(m_v,\frac{\partial\bar{l}}{\partial\theta_0}\right),0,\gamma,n\,\mathrm{cov}_g\left(\frac{\partial v^{(\mathrm{A})}}{\partial\theta_0},\frac{\partial\bar{l}}{\partial\theta_0}\right)\right\}$$

$$+12\alpha^{(\mathrm{A})}_{\mathrm{ML}2}(-2)^2\underset{(\mathrm{C})}{\Bigg[}\ \{n\,\mathrm{cov}_g(\bar{l}_0,m_v)\}^2,\ n\,\mathrm{cov}_g(\bar{l}_0,m_v)\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial\theta_0}\right),0,$$

$$\left\{\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial\theta_0}\right)\right\}^2,n\,\mathrm{cov}_g\left(\bar{l}_0,\frac{\partial v^{(\mathrm{A})}}{\partial\theta_0}\right)\mathrm{E}_g\left(l_{0j}\frac{\partial l_j}{\partial\theta_0}\right)\ \underset{(\mathrm{C})}{\Bigg]}\ \underset{(\mathrm{B})}{\Bigg]}\mathbf{v}^{(2)}(\alpha^{(\mathrm{A})}_{\mathrm{ML}2})^{-3/2}$$

$$
\begin{aligned}
&= 12 \Bigg[_{(\mathrm{B})} \; [\, (\kappa_4 + 2)^2 n \,\mathrm{avar}_g (m_v) + 16(\kappa_4 + 2)\{n \,\mathrm{cov}_g (\bar{l}_0, m_v)\}^2 \;] \frac{3}{8} (\kappa_4 + 2)^{-4} \\
&\quad + \left\{ (\kappa_4 + 2)^2 n \,\mathrm{cov}_g \left(m_v, \frac{\partial \bar{l}}{\partial \theta_0} \right) + 16(\kappa_4 + 2) n \,\mathrm{cov}_g (\bar{l}_0, m_v) \mathrm{E}_g \left(l_{0j} \frac{\partial l_j}{\partial \theta_0} \right) \right\} \\
&\qquad \times \{-3(\kappa_4 + 2)^{-4} \kappa_3 \sigma\} \\
&\quad + \left[(\kappa_4 + 2)^2 \gamma + 16(\kappa_4 + 2) \left\{ \mathrm{E}_g \left(l_{0j} \frac{\partial l_j}{\partial \theta_0} \right) \right\}^2 \right] \\
&\qquad\qquad \times \{-2(\kappa_4 + 2)^{-3} + 6(\kappa_4 + 2)^{-4} \kappa_3^2\} \sigma^2 \\
&\quad + \left\{ (\kappa_4 + 2)^2 n \,\mathrm{cov}_g \left(\frac{\partial v^{(\mathrm{A})}}{\partial \theta_0}, \frac{\partial \bar{l}}{\partial \theta_0} \right) \right. \\
&\qquad \left. + 16(\kappa_4 + 2) n \,\mathrm{cov}_g \left(\bar{l}_0, \frac{\partial v^{(\mathrm{A})}}{\partial \theta_0} \right) \mathrm{E}_g \left(l_{0j} \frac{\partial l_j}{\partial \theta_0} \right) \right\} \left\{ -\frac{\sigma^2}{2} (\kappa_4 + 2)^{-3} \right\} \; \Bigg]_{(\mathrm{B})} \\
&\quad + O(n^{-1})
\end{aligned}
$$

$$=12\Big[_{(B)}\ \{ (\kappa_4+2)^{-2}(\kappa_8+24\kappa_6+56\kappa_5\kappa_3+34\kappa_4^2+152\kappa_4+240\kappa_3^2+56)$$

$$+4(\kappa_4+2)^{-3}(\kappa_6+12\kappa_4+10\kappa_3^2+8)^2\ \}\frac{3}{8}$$

$$+\Big[(\kappa_4+2)^{-2}(\kappa_5+8\kappa_3)\frac{1}{\sigma}$$

$$+16(\kappa_4+2)^{-3}\left\{-\frac{1}{2}(\kappa_6+12\kappa_4+10\kappa_3^2+8)\right\}\left(-\frac{\kappa_3}{2\sigma}\right)\Bigg](-3\kappa_3\sigma)$$

$$+\left\{(\kappa_4+2)^{-2}\frac{1}{\sigma^2}+16(\kappa_4+2)^{-3}\frac{\kappa_3^2}{4\sigma^2}\right\}\{-2(\kappa_4+2)+6\kappa_3^2\}\sigma^2$$

$$+\left\{-\frac{4}{\sigma^2}+16(\kappa_4+2)^{-2}\frac{2}{\sigma}(\kappa_5+8\kappa_3)\left(-\frac{\kappa_3}{2\sigma}\right)\right\}\left(-\frac{\sigma^2}{2}\right)\ \Bigg]_{(B)}+O(n^{-1})$$

$$=12\Big[_{(B)}\ (\kappa_4+2)^{-3}\left\{\frac{3}{2}(\kappa_6+12\kappa_4+10\kappa_3^2+8)^2\right.$$

$$\left.-12(\kappa_6+12\kappa_4+10\kappa_3^2+8)\kappa_3^2+24\kappa_3^4\right\}$$

$$+(\kappa_4+2)^{-2}\left\{\frac{3}{8}(\kappa_8+24\kappa_6+56\kappa_5\kappa_3+34\kappa_4^2+152\kappa_4+240\kappa_3^2+56)\right.$$

$$\left.-3(\kappa_5+8\kappa_3)\kappa_3+6\kappa_3^2-8\kappa_3^2+8(\kappa_5+8\kappa_3)\kappa_3\right\}$$

$$-2(\kappa_4+2)^{-1}+2\ \Big]_{(B)}+O(n^{-1})$$

$$=12\underset{(\mathrm{B})}{\Bigg[}\ (\kappa_4+2)^{-3}\frac{3}{2}(\kappa_6+12\kappa_4+6\kappa_3^2+8)^2$$

$$+\ (\kappa_4+2)^{-2}\left\{\frac{3}{8}(\kappa_8+24\kappa_6+34\kappa_4^2+152\kappa_4+56)+26\kappa_5\kappa_3+128\kappa_3^2\right\}$$

$$-2(\kappa_4+2)^{-1}+2\quad\underset{(\mathrm{B})}{\Bigg]}+O(n^{-1})$$

$$\left(=12\left(\frac{1}{8}\times\frac{3}{2}\times64+\frac{1}{4}\times\frac{3}{8}\times56-1+2\right)=144+63+12=219\ \ \text{under normality}\right),$$

(vii) the sixth term in $\underset{(\mathrm{A})}{[}\ \cdot\ \underset{(\mathrm{A})}{]}$ of (S6.16) is

$$[4n\mathrm{E}_g(\bar{l}^{(2)}_{\mathrm{ML}})\alpha^{(\mathrm{A})}_{\mathrm{ML3}}+6\alpha^{(\mathrm{A})}_{\mathrm{ML2}}\alpha^{(\mathrm{A})}_{\mathrm{ML}\Delta2}+6\alpha^{(\mathrm{A})}_{\mathrm{ML2}}\{n\mathrm{E}_g(\bar{l}^{(2)}_{\mathrm{ML}})\}^2](\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-2}$$

$$=[\ 4(-1)(\kappa_6+12\kappa_4+4\kappa_3^2+8)+6(\kappa_4+2)\{-2(\kappa_4+1)\}$$

$$+6(\kappa_4+2)(-1)^2\ \ ](\kappa_4+2)^{-2}$$

$$=-\{4\kappa_6+(48+36-6)\kappa_4+12\kappa_4^2+16\kappa_3^2+32+24-12\}(\kappa_4+2)^{-2}$$

$$=-(4\kappa_6+78\kappa_4+12\kappa_4^2+16\kappa_3^2+44)(\kappa_4+2)^{-2}$$

$(=-44/4=-11$ under normality),

(viii) the sum of the seventh, eighth and ninth terms in $\underset{(\mathrm{A})}{[}\ \cdot\ \underset{(\mathrm{A})}{]}$ of (S6.16) is

$$-4\{\alpha^{(\mathrm{A})}_{(t)\mathrm{ML1}}-2q(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}\}\alpha^{(\mathrm{A})}_{(t)\mathrm{ML3}}$$

$$-6\{\alpha^{(\mathrm{A})}_{(t)\mathrm{ML}\Delta2}-4qn\mathrm{E}_g(\bar{l}^{(1)}_{\mathrm{ML}}\mathbf{m}^{(1)}_v{}')\mathbf{v}^{(1)}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}\}-6\{\alpha^{(\mathrm{A})}_{(t)\mathrm{ML1}}-2q(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}\}^2$$

$$=-4\left\{(\kappa_4+2)^{-1/2}-\frac{1}{2}(\kappa_4+2)^{-3/2}(\kappa_6+12\kappa_4+6\kappa_3^2+8)-2q(\kappa_4+2)^{-1/2}\right\}$$

$$\times(-2)(\kappa_4+2)^{-3/2}(\kappa_6+12\kappa_4+7\kappa_3^2+8)$$

$$-6\left[\alpha^{(\mathrm{A})}_{(t)\mathrm{ML}\Delta2}-4q\left\{-\frac{1}{2}(\kappa_4+2)^{-3/2}(\kappa_6+12\kappa_4+6\kappa_3^2+8)\right\}(\kappa_4+2)^{-1/2}\right]$$

$$
\begin{aligned}
&-6\left\{(\kappa_4+2)^{-1/2}-\frac{1}{2}(\kappa_4+2)^{-3/2}(\kappa_6+12\kappa_4+6\kappa_3^2+8)-2q(\kappa_4+2)^{-1/2}\right\}^2 \\
&=-8\left\{1+\frac{1}{2}(\kappa_4+2)^{-1}(\kappa_6+12\kappa_4+6\kappa_3^2+8)\right\} \\
&\qquad\times(\kappa_4+2)^{-2}(\kappa_6+12\kappa_4+7\kappa_3^2+8) \\
&\quad-6\{\ \alpha_{(t)\mathrm{ML}\Delta 2}^{(\mathrm{A})}+2(\kappa_4+2)^{-2}(\kappa_6+12\kappa_4+6\kappa_3^2+8)\ \} \\
&\quad-6\left\{1+\frac{1}{2}(\kappa_4+2)^{-1}(\kappa_6+12\kappa_4+6\kappa_3^2+8)\right\}^2(\kappa_4+2)^{-1} \\
&=-6\alpha_{(t)\mathrm{ML}\Delta 2}^{(\mathrm{A})}-(8+12+6)(\kappa_4+2)^{-2}(\kappa_6+12\kappa_4+6\kappa_3^2+8) \\
&\quad-\{\ 8(\kappa_4+2)^{-2}\kappa_3^2+4(\kappa_4+2)^{-3}(\kappa_6+12\kappa_4+6\kappa_3^2+8)\kappa_3^2\ \} \\
&\quad-6(\kappa_4+2)^{-1}-\left(4+\frac{3}{2}\right)(\kappa_4+2)^{-3}(\kappa_6+12\kappa_4+6\kappa_3^2+8)^2 \\
&=-6\{\ 2-2(\kappa_4+2)^{-1}+(\kappa_4+2)^{-2}(-\kappa_6+8\kappa_5\kappa_3+2\kappa_4^2-4\kappa_4+50\kappa_3^2) \\
&\qquad+(\kappa_4+2)^{-3}(7/4)(\kappa_6+12\kappa_4+6\kappa_3^2+8)^2\ \} \\
&\quad-26(\kappa_4+2)^{-2}(\kappa_6+12\kappa_4+6\kappa_3^2+8)-8(\kappa_4+2)^{-2}\kappa_3^2-6(\kappa_4+2)^{-1} \\
&\quad-(11/2)(\kappa_4+2)^{-3}(\kappa_6+12\kappa_4+6\kappa_3^2+8)^2 \\
&\quad-4(\kappa_4+2)^{-3}(\kappa_6+12\kappa_4+6\kappa_3^2+8)\kappa_3^2 \\
&=-12+6(\kappa_4+2)^{-1}+(\kappa_4+2)^{-2}\{\ (6-26)\kappa_6-48\kappa_5\kappa_3-12\kappa_4^2 \\
&\qquad\qquad+(24-26\times 12)\kappa_4+(-6\times 50-26\times 6-8)\kappa_3^2-26\times 8\ \} \\
&\quad-\left(\frac{21}{2}+\frac{11}{2}\right)(\kappa_4+2)^{-3}(\kappa_6+12\kappa_4+6\kappa_3^2+8)^2 \\
&\quad-4(\kappa_4+2)^{-3}(\kappa_6+12\kappa_4+6\kappa_3^2+8)\kappa_3^2 \\
&=-12+6(\kappa_4+2)^{-1} \\
&\qquad+(\kappa_4+2)^{-2}(-20\kappa_6-48\kappa_5\kappa_3-12\kappa_4^2-288\kappa_4-464\kappa_3^2-208) \\
&\quad-16(\kappa_4+2)^{-3}(\kappa_6+12\kappa_4+6\kappa_3^2+8)^2 \\
&\quad-4(\kappa_4+2)^{-3}(\kappa_6+12\kappa_4+6\kappa_3^2+8)\kappa_3^2
\end{aligned}
$$

$(=-12+3-52-128=-189$ under normality).

Consequently,

$$
\begin{aligned}
&\kappa_{g4}(t_{\mathrm{ML}}^{(\mathrm{A})})\\
&= n^{-1}\Bigg[\ (\kappa_4+2)^{-2}(\kappa_8+24\kappa_6+32\kappa_5\kappa_3+32\kappa_4^2+144\kappa_4+96\kappa_3^2+48)\Bigg._{(\mathrm{A})}\\
&\quad -12(\kappa_4+2)^{-2}(\kappa_8+24\kappa_6+52\kappa_5\kappa_3+32\kappa_4^2+144\kappa_4+208\kappa_3^2+48)\\
&\quad -8(\kappa_4+2)^{-3}\{\,(\kappa_6+12\kappa_4+8\kappa_3^2+8)^2-4\kappa_3^4\,\}\\
&\qquad +24(\kappa_4+2)^{-2}(\kappa_6+12\kappa_4+14\kappa_3^2+2\kappa_5\kappa_3+8)\\
&\qquad +48(\kappa_4+2)^{-3}\kappa_3^2(\kappa_6+12\kappa_4+6\kappa_3^2+8)\\
&\quad +\frac{9}{2}(\kappa_4+2)^{-2}(\kappa_8+24\kappa_6+48\kappa_5\kappa_3+34\kappa_4^2+152\kappa_4+192\kappa_3^2+56)\\
&\quad +18(\kappa_4+2)^{-3}(\kappa_6+12\kappa_4+6\kappa_3^2+8)^2\\
&\quad +18(\kappa_4+2)^{-3}(\kappa_6+12\kappa_4+6\kappa_3^2+8)^2\\
&\quad +(\kappa_4+2)^{-2}\left\{\frac{9}{2}(\kappa_8+24\kappa_6+34\kappa_4^2+152\kappa_4+56)+312\kappa_5\kappa_3+1536\kappa_3^2\right\}\\
&\quad -24(\kappa_4+2)^{-1}+24\\
&\quad -(\kappa_4+2)^{-2}(4\kappa_6+78\kappa_4+12\kappa_4^2+16\kappa_3^2+44)\\
&\quad -12+6(\kappa_4+2)^{-1}\\
&\qquad +(\kappa_4+2)^{-2}(-20\kappa_6-48\kappa_5\kappa_3-12\kappa_4^2-288\kappa_4-464\kappa_3^2-208)\\
&\quad -16(\kappa_4+2)^{-3}(\kappa_6+12\kappa_4+6\kappa_3^2+8)^2\\
&\quad -4(\kappa_4+2)^{-3}(\kappa_6+12\kappa_4+6\kappa_3^2+8)\kappa_3^2\ \Bigg]_{(\mathrm{A})}+O(n^{-2})
\end{aligned}
\tag{S6.17}
$$

$$
\begin{aligned}
&= n^{-1} \underset{\text{(A)}}{\Bigg[} \ 12 - 18(\kappa_4 + 2)^{-1} \\
&\quad + (\kappa_4 + 2)^{-2} \{ \ (1 - 12 + 9)\kappa_8 + (24 - 12 \times 24 + 24 + 9 \times 24 - 4 - 20)\kappa_6 \\
&\qquad + (32 - 12 \times 52 + 48 + 9 \times 24 + 312 - 48)\kappa_5\kappa_3 \\
&\qquad + (32 - 12 \times 32 + 9 \times 34 - 12 - 12)\kappa_4^2 \\
&\qquad + (144 - 12 \times 144 + 24 \times 12 + 9 \times 152 - 78 - 288)\kappa_4 \\
&\qquad + (96 - 12 \times 208 + 24 \times 14 + 9 \times 96 + 1536 - 16 - 464)\kappa_3^2 \\
&\qquad + 48 - 12 \times 48 + 24 \times 8 + 9 \times 56 - 44 - 208 \ \} \\
&\quad + (\kappa_4 + 2)^{-3} \{ -8(\kappa_6 + 12\kappa_4 + 8\kappa_3^2 + 8)^2 + 32\kappa_3^4 \\
&\qquad + (48 - 4)(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)\kappa_3^2 \\
&\qquad + (36 - 16)(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 \ \} \ \underset{\text{(A)}}{\Bigg]} + O(n^{-2})
\end{aligned}
$$

$$
\begin{aligned}
&= n^{-1}[\ 12 - 18(\kappa_4 + 2)^{-1} \\
&\quad + (\kappa_4 + 2)^{-2}(-2\kappa_8 - 48\kappa_6 - 64\kappa_5\kappa_3 - 70\kappa_4^2 - 294\kappa_4 - 144\kappa_3^2 - 84) \\
&\quad + (\kappa_4 + 2)^{-3}\{ \ 12(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 + 12(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)\kappa_3^2 \ \} \ ] \\
&\qquad + O(n^{-2}) \\
&= n^{-1}\alpha_{(t)\mathrm{ML4}}^{(\mathrm{A})} + O(n^{-2})
\end{aligned}
$$

($= 12 - 9 - 21 + 96 = 78$ under normality).

S6.2.3 A result for estimation of $-2\overline{l}_0^*$

$$
\begin{aligned}
& n\,\mathrm{acov}_g\left\{n^{-1}\mathrm{AIC}_{\mathrm{ML}},\ \hat{\alpha}^{(\mathrm{A})}_{(t)\mathrm{ML1}} + \frac{\hat{\alpha}^{(\mathrm{A})}_{(t)\mathrm{ML3}}}{6}(z^2_{\tilde{\alpha}} - 1)\right\} \\
& = n\,\mathrm{acov}_g\left\{\ n^{-1}\mathrm{AIC}_{\mathrm{ML}},\ \hat{\alpha}^{(\mathrm{A})}_{\mathrm{ML1}}(\hat{\alpha}^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2} + \hat{\alpha}^{(\mathrm{A})}_{(\Delta t)\mathrm{ML1}} z^2_{\tilde{\alpha}} \right. \\
& \qquad\qquad \left. + \frac{1}{6}\hat{\alpha}^{(\mathrm{A})}_{\mathrm{ML3}}(\hat{\alpha}^{(\mathrm{A})}_{\mathrm{ML2}})^{-3/2}(z^2_{\tilde{\alpha}} - 1)\right\},
\end{aligned}
\tag{S6.18}
$$

(i) where the first term on the right-hand side of (S6.18) is

$$
\begin{aligned}
& n\,\mathrm{acov}_g\{n^{-1}\mathrm{AIC}_{\mathrm{ML}},\ \hat{\alpha}^{(\mathrm{A})}_{\mathrm{ML1}}(\hat{\alpha}^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}\} \\
& = n\,\mathrm{acov}_g\{\bar{l}^{(1)}_{\mathrm{ML}}, (\hat{\alpha}^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}\} = n\mathrm{E}_g(\bar{l}^{(1)}_{\mathrm{ML}}\mathbf{m}^{(1)}_v{}')\mathbf{v}^{(1)} \\
& = -\frac{1}{2}(\kappa_4 + 2)^{-3/2}(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8),
\end{aligned}
$$

(ii) the second term on the right-hand side of (S6.18) is

$$
\begin{aligned}
& n\,\mathrm{acov}_g\{n^{-1}\mathrm{AIC}_{\mathrm{ML}},\ \hat{\alpha}^{(\mathrm{A})}_{(\Delta t)\mathrm{ML1}} z^2_{\tilde{\alpha}}\} \\
& = n\,\mathrm{acov}_g\{\bar{l}^{(1)}_{\mathrm{ML}}, n\widehat{\mathrm{E}_g(\bar{l}^{(1)}_{\mathrm{ML}}\mathbf{m}^{(1)}_v{}')\mathbf{v}^{(1)}}\} z^2_{\tilde{\alpha}} \\
& = n\,\mathrm{acov}_g[\ \bar{l}^{(1)}_{\mathrm{ML}}, (\hat{\alpha}^{(\mathrm{A})}_{\mathrm{ML2}})^{-3/2}\{4\widehat{\kappa_{g3}(l_{0j})} + 2\hat{\kappa}_3^2\}\ ] z^2_{\tilde{\alpha}}
\end{aligned}
$$

with

$$
\begin{aligned}
& 4\widehat{\kappa_{g3}(l_{0j})} \equiv 4n^{-1}\sum_{j=1}^{n}\left\{\hat{l}_{\mathrm{ML}j} - \left(-\frac{1}{2} - \frac{1}{2}\log(2\pi\sigma^2)\right)\right\}^3 \\
& \qquad\qquad (\hat{l}_{\mathrm{ML}j} = -\{(x_j - \bar{x})^2/(2\sigma^2)\} - (1/2)\log(2\pi\sigma^2)) \\
& = -\frac{1}{2\sigma^6}n^{-1}\sum_{j=1}^{n}\{(x_j - \bar{x})^2 - \sigma^2\}^3 \\
& = -\frac{1}{2\sigma^6}\left[n^{-1}\sum_{j=1}^{n}\{(x_j - \mu_0)^2 - \sigma^2\}^3\right. \\
& \qquad \left. - 6n^{-1}\sum_{j=1}^{n}\{(x_j - \mu_0)^2 - \sigma^2\}^2(x_j - \mu_0)(\bar{x} - \mu_0)\right] + O_p(n^{-1})
\end{aligned}
$$

$$= -\frac{1}{2}\left[\overline{z^6} - 3\overline{z^4} + 3\overline{z^2} - 1 - 6\{\overline{z^5} - 2\overline{z^3} + (\bar{z})_{O_p(n^{-1/2})}\}(\bar{z})_{O_p(n^{-1/2})}\right] + O_p(n^{-1})$$

$$\left(\overline{z^k} \equiv n^{-1}\sum_{j=1}^{n}\left(\frac{x_j-\mu_0}{\sigma}\right)^k\right)$$

and

$$2\hat{\kappa}_3^2 = 2\left\{n^{-1}\sum_{j=1}^{n}\left(\frac{x_j-\bar{x}}{\sigma}\right)^3\right\}^2 = 2\left\{n^{-1}\sum_{j=1}^{n}\left(\frac{x_j-\mu_0}{\sigma}\right)^3\right\}^2$$

$$-12n^{-1}\sum_{j=1}^{n}\left(\frac{x_j-\mu_0}{\sigma}\right)^3 n^{-1}\sum_{k=1}^{n}\left(\frac{x_k-\mu_0}{\sigma}\right)^2 \frac{\bar{x}-\mu_0}{\sigma} + O_p(n^{-1})$$

$$= 2\overline{z^3}^2 - 12\overline{z^3}\ \overline{z^2}\,(\bar{z})_{O_p(n^{-1/2})} + O_p(n^{-1}),$$

that is

$$n\,\mathrm{acov}_g[\ \bar{l}_{\mathrm{ML}}^{(1)}, (\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\{4\widehat{\kappa_{g3}(l_{0j})} + 2\hat{\kappa}_3^2\}\ ]z_{\tilde{\alpha}}^2$$

$$= n\,\mathrm{acov}_g\{-2\bar{l}_0,\ 3(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\}\{4\kappa_{g3}(l_{0j}) + 2\kappa_3^2\}z_{\tilde{\alpha}}^2$$

$$+ n\,\mathrm{acov}_g\{-2\bar{l}_0,\ 4\widehat{\kappa_{g3}(l_{0j})} + 2\hat{\kappa}_3^2\}(\kappa_4+2)^{-3/2}\, z_{\tilde{\alpha}}^2$$

$$= 3(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1}\left\{-\frac{1}{2}(\kappa_4+2)^{-3/2}(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)\right\}$$

$$\times\left\{-\frac{1}{2}(\kappa_6 + 12\kappa_4 + 10\kappa_3^2 + 8) + 2\kappa_3^2\right\}z_{\tilde{\alpha}}^2$$

$$+ n\,\mathrm{acov}_g\left[\overline{z^2},\ -\frac{1}{2}\left\{\overline{z^6} - 3\overline{z^4} + 3\overline{z^2} - 1 - 6(\overline{z^5} - 2\overline{z^3} + \bar{z})\bar{z}\right\}\right.$$

$$\left. + 2\overline{z^3}^2 - 12\overline{z^3}\ \overline{z^2}\ \bar{z}\quad\right](\kappa_4+2)^{-3/2} z_{\tilde{\alpha}}^2,$$

$$= \frac{3}{4}(\kappa_4 + 2)^{-5/2}(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 z_{\tilde{\alpha}}^2$$

$$+ \left[-\frac{1}{2}\{(\sigma_8 - \sigma_6) - 3(\sigma_6 - \sigma_4) + 3(\sigma_4 - 1)\} + 3(\sigma_5 - 2\sigma_3)\sigma_3 \right.$$

$$\left. + 4\sigma_3(\sigma_5 - \sigma_3) - 12\sigma_3^2 \right](\kappa_4 + 2)^{-3/2} z_{\tilde{\alpha}}^2$$

$$= \left[\frac{3}{4}(\kappa_4 + 2)^{-5/2}(\kappa_6 + 12\kappa_4 + 6\kappa_3^2 + 8)^2 \right.$$

$$\left. + (\kappa_4 + 2)^{-3/2} \left\{ -\frac{1}{2}(\sigma_8 - 4\sigma_6 + 6\sigma_4 - 3) + 7\sigma_5\sigma_3 - 22\sigma_3^2 \right\} \right] z_{\tilde{\alpha}}^2,$$

where

$$\sigma_k \equiv \mathrm{E}_g(z^k),\ \sigma_2 = \kappa_2 = 1, \sigma_3 = \kappa_3, \sigma_4 = \kappa_4 + 3,$$

$$\mathrm{cov}_g(\overline{z^2},\ \bar{z}^2) = n^{-3}\mathrm{E}_g\left\{ \sum_{j=1}^{n}(z_j^2 - 1)\sum_{k=1}^{n} z_k \sum_{l^*=1}^{n} z_{l^*} \right\}$$

$$= n^{-3}(n\sigma_4 + n^2 - n - n^2) = n^{-2}(\sigma_4 - 1) = O(n^{-2}).$$

(iii) the third term on the right-hand side of (S6.18)

Using $\hat{\alpha}_{\mathrm{ML3}}^{(\mathrm{A})} \equiv \hat{\kappa}_6 + 12\hat{\kappa}_4 + 4\hat{\kappa}_3^2 + 8 = \hat{\sigma}_6 - 3\hat{\sigma}_4 - 6\hat{\sigma}_3^2 + 2$ with $\hat{\sigma}_k \equiv n^{-1}\sum_{j=1}^{n}\{(x_j - \bar{x})/\sigma\}^k$, we have

$$n\,\mathrm{acov}_g\left\{ n^{-1}\mathrm{AIC}_{\mathrm{ML}},\ \frac{1}{6}\hat{\alpha}_{\mathrm{ML3}}^{(\mathrm{A})}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}(z_{\tilde{\alpha}}^2 - 1) \right\}$$

$$= \left[n\,\mathrm{acov}_g\left\{ \overline{z^2},\ \overline{z^6} - 3\overline{z^4} - 6\overline{z^3}^2 - 6(\overline{z^5} - 2\overline{z^3} - 6\overline{z^3}\,\overline{z^2})\bar{z} \right\}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2} \right.$$

$$\left. + n\,\mathrm{acov}_g\{\ \bar{l}_{\mathrm{ML}}^{(1)},\ 3(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\}(\kappa_6 + 12\kappa_4 + 4\kappa_3^2 + 8) \right]\frac{z_{\tilde{\alpha}}^2 - 1}{6}$$

$$
\begin{aligned}
=\Big[\ & (\kappa_4+2)^{-3/2}\{\ (\sigma_8-\sigma_6)-3(\sigma_6-\sigma_4)-12\sigma_3(\sigma_5-\sigma_3) \\
&\qquad\qquad -6(\sigma_5-2\sigma_3-6\sigma_3)\sigma_3\ \} \\
&+3(\kappa_4+2)^{-1}\left\{-\frac{1}{2}(\kappa_4+2)^{-3/2}(\kappa_6+12\kappa_4+6\kappa_3^2+8)\right\} \\
&\qquad\qquad \times(\kappa_6+12\kappa_4+4\kappa_3^2+8)\ \Big]\frac{z_{\tilde{\alpha}}^2-1}{6}
\end{aligned}
$$

$$
\begin{aligned}
=\Big[\ & (\kappa_4+2)^{-3/2}(\ \sigma_8-4\sigma_6+3\sigma_4-18\sigma_3\sigma_5+60\sigma_3^2) \\
&-\frac{3}{2}(\kappa_4+2)^{-5/2}(\kappa_6+12\kappa_4+6\kappa_3^2+8)(\kappa_6+12\kappa_4+4\kappa_3^2+8)\ \Big]\frac{z_{\tilde{\alpha}}^2-1}{6}.
\end{aligned}
$$

Consequently,

$$
n\,\mathrm{acov}_g\left\{n^{-1}\mathrm{AIC}_{\mathrm{ML}},\ \hat{\alpha}_{(t)\mathrm{ML}1}^{(\mathrm{A})}+\frac{\hat{\alpha}_{(t)\mathrm{ML}3}^{(\mathrm{A})}}{6}(z_{\tilde{\alpha}}^2-1)\right\} \qquad \text{(S6.19)}
$$

is the sum of the values in (i), (ii) and (iii) given above (under normality, (S6.19) is 0 since $\hat{\alpha}_{(t)\mathrm{ML}1}^{(\mathrm{A})}=\alpha_{(t)\mathrm{ML}1}^{(\mathrm{A})}=2^{-1/2}-2^{1/2}$ and $\hat{\alpha}_{(t)\mathrm{ML}3}^{(\mathrm{A})}=\alpha_{(t)\mathrm{ML}3}^{(\mathrm{A})}=-2^{5/2}$ are fixed values).

S6.2.4 Asymptotic cumulants of $n^{-1}\mathrm{AIC}_{\mathrm{ML}}$ after studentization for estimation of $-2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{ML}}^{*})$

$$
t_{\mathrm{ML}}^{(\mathrm{A})*}=\frac{n^{1/2}\{n^{-1}\mathrm{AIC}_{\mathrm{ML}}+2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{ML}}^{*})\}}{(\hat{v}_{\mathrm{ML}}^{(\mathrm{A})})^{1/2}}.
$$

$$
\begin{aligned}
\kappa_{g1}(t_{\mathrm{ML}}^{(\mathrm{A})*}) &= n^{-1/2}\{\alpha_{(t)\mathrm{ML1}}^{(\mathrm{A})} + \lambda^{-1}\gamma(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\} + O(n^{-3/2}) \\
&= n^{-1/2}\left\{(\kappa_4+2)^{-1/2} - \frac{1}{2}(\kappa_4+2)^{-3/2}(\kappa_6+12\kappa_4+6\kappa_3^2+8)\right. \\
&\qquad\qquad \left. -(\kappa_4+2)^{-1/2}\right\} + O(n^{-3/2}) \\
&= -n^{-1/2}\frac{1}{2}(\kappa_4+2)^{-3/2}(\kappa_6+12\kappa_4+6\kappa_3^2+8) + O(n^{-3/2}) \\
&= n^{-1/2}\alpha_{(t)\mathrm{ML1}}^{(\mathrm{A})*} + O(n^{-3/2}) \ \ (\alpha_{(t)\mathrm{ML1}}^{(\mathrm{A})*} = \alpha_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})})
\end{aligned}
\tag{S6.20}
$$

(under normality $\alpha_{(t)\mathrm{ML1}}^{(\mathrm{A})*} = \alpha_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})} = -2^{1/2}$),

$$
\begin{aligned}
\kappa_{g2}(t_{\mathrm{ML}}^{(\mathrm{A})*}) &= 1 + n^{-1}\{\alpha_{(t)\mathrm{ML}\Delta2}^{(\mathrm{A})} + 2\lambda^{-1}\gamma(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} n\mathrm{E}_g(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)}\} + O(n^{-2}) \\
&= 1 + n^{-1}\{\,\alpha_{(t)\mathrm{ML}\Delta2}^{(\mathrm{A})} + (\kappa_4+2)^{-2}(\kappa_6+12\kappa_4+6\kappa_3^2+8)\,\} + O(n^{-2}) \\
&= 1 + n^{-1}\alpha_{(t)\mathrm{ML}\Delta2}^{(\mathrm{A})*} + O(n^{-2}),
\end{aligned}
$$

(under normality $\alpha_{(t)\mathrm{ML}\Delta2}^{(\mathrm{A})*} = \alpha_{(\Delta t)\mathrm{ML}\Delta2}^{(\mathrm{A})} + 2$),

$$
\begin{aligned}
\kappa_{g3}(t_{\mathrm{ML}}^{(\mathrm{A})*}) &= n^{-1/2}\alpha_{(t)\mathrm{ML3}}^{(\mathrm{A})} + O(n^{-3/2}) \ \ (\alpha_{(t)\mathrm{ML3}}^{(\mathrm{A})*} = \alpha_{(t)\mathrm{ML3}}^{(\mathrm{A})}), \\
\kappa_{g4}(t_{\mathrm{ML}}^{(\mathrm{A})*}) &= n^{-1}\alpha_{(t)\mathrm{ML4}}^{(\mathrm{A})} + O(n^{-2}) \ \ (\alpha_{(t)\mathrm{ML4}}^{(\mathrm{A})*} = \alpha_{(t)\mathrm{ML4}}^{(\mathrm{A})}).
\end{aligned}
$$

S6.2.5 A result for estimation of $-2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{ML}}^{*})$

$$
\begin{aligned}
& n\,\mathrm{acov}_g\left\{n^{-1}\mathrm{AIC}_{\mathrm{ML}},\ \hat{\alpha}_{(t)\mathrm{ML1}}^{(\mathrm{A})*} + \frac{\hat{\alpha}_{(t)\mathrm{ML3}}^{(\mathrm{A})}}{6}(z_{\tilde{\alpha}}^2 - 1)\right\} \\
&= n\,\mathrm{acov}_g\left\{n^{-1}\mathrm{AIC}_{\mathrm{ML}},\ \hat{\alpha}_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})} z_{\tilde{\alpha}}^2 + \frac{1}{6}\hat{\alpha}_{\mathrm{ML3}}^{(\mathrm{A})}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}(z_{\tilde{\alpha}}^2 - 1)\right\}.
\end{aligned}
\tag{S6.21}
$$

This reduces to the case of Subsection S6.2.3, where the first term of (i) for (S6.18) is zero. So, (S6.21) is equal to the sum of the second and third terms given in (ii) and (iii) for (S6.18), respectively.

S6.2.6 Higher-order bias correction of $n^{-1}\mathrm{AIC}_{\mathrm{ML}}$

Since the correction term of $n^{-1}\mathrm{AIC}_{\mathrm{ML}}$ i.e., $2n^{-1}$ is exact even under non-normality, the higher-order correction is unnecessary.

S6.3 $n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(j)}\,(j=1,\,2)$

Since $\lambda=\hat{\lambda}=-1/\sigma^2$ and $\gamma=\hat{\gamma}=1/\sigma^2$,

$$\begin{aligned} n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(j)} &= -2\hat{\bar{l}}_{\mathrm{ML}}+n^{-1}2(-\hat{\lambda}^{-1}\hat{\gamma})=-2\hat{\bar{l}}_{\mathrm{ML}}+n^{-1}2 \\ &= n^{-1}\mathrm{AIC}_{\mathrm{ML}}\ \ (j=1,\,2). \end{aligned} \tag{S6.22}$$

That is, the results in Subsection S6.2 for $n^{-1}\mathrm{AIC}_{\mathrm{ML}}$ also hold for $n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(j)}\,(j=1,\,2)$.

S7. Example 3: The Bernoulli distribution with the WSE of the logit under correct model specification

S7.1 Preliminary results

$$\Pr(x^* = x \,|\, \theta) = \pi^x (1-\pi)^{1-x} \quad (x = 0, 1), \;\; \pi = \frac{1}{1+\exp(-\theta)},$$

$$l = \log\left\{\prod_{j=1}^{n} \pi^{x_j} (1-\pi)^{1-x_j}\right\} = \sum_{j=1}^{n} l_j, \;\; \overline{l} = n^{-1} l,$$

$$l_j = x_j \log\frac{\pi}{1-\pi} + \log(1-\pi) = x_j\theta + \log(1-\pi),$$

$$l_{0j} = x_j\theta_0 + \log(1-\pi_0), \; \pi_0 = \frac{1}{1+\exp(-\theta_0)}, \; \overline{l}_0 = n^{-1}\sum_{j=1}^{n} l_{0j},$$

$$l_{\mathrm{W}}^{(\mathrm{e})} = \log\left[\left\{\prod_{j=1}^{n} \pi^{x_j} (1-\pi)^{1-x_j}\right\}\{\pi(1-\pi)\}^{a/2}\right], \tag{S7.1}$$

$$\begin{aligned}\overline{l}_{\mathrm{W}}^{(\mathrm{e})} &= n^{-1} l_{\mathrm{W}}^{(\mathrm{e})} \\ &= (\overline{x} + n^{-1} 0.5a)\log\pi + (1-\overline{x} + n^{-1} 0.5a)\log(1-\pi),\end{aligned}$$

$$\overline{l}_{0\mathrm{W}}^{(\mathrm{e})} = (\overline{x} + n^{-1} 0.5a)\log\pi_0 + (1-\overline{x} + n^{-1} 0.5a)\log(1-\pi_0),$$

where the superscript (e) indicates that the quantity is for estimation of θ_0 by the weighted score method and *a* is the sum of the pseudocounts for two categories,

$$\overline{l}_0^* = \mathrm{E}_f(l_{0j}) = \mathrm{E}_f(\overline{l}_0) = \pi_0\theta_0 + \log(1-\pi_0),$$

$$\frac{\partial \overline{l}}{\partial \theta_0} = \left(\frac{\overline{x}}{\pi_0} - \frac{1-\overline{x}}{1-\pi_0}\right)\pi_0(1-\pi_0) = \overline{x} - \pi_0, \; \hat{\theta}_{\mathrm{ML}} = \log\frac{\overline{x}}{1-\overline{x}},$$

$$\hat{\pi}_{\mathrm{ML}} = \overline{x}, \; \boldsymbol{\Lambda} = \lambda = \frac{\partial^2 \overline{l}}{\partial \theta_0^2} = \frac{\partial^2 l_j}{\partial \theta_0^2} = -\pi_0(1-\pi_0) = -\overline{i}_0,$$

$$\mathbf{\Gamma} = \gamma = n\mathrm{E}_f \left\{ \left(\frac{\partial \bar{l}}{\partial \theta_0} \right)^2 \right\} = n\,\mathrm{var}_f(\bar{x}) = \pi_0(1-\pi_0) = \bar{i}_0,$$

$$\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma}) = \lambda^{-1}\gamma = -1,$$

$$l_{0j} - \bar{l}_0^* = (x_j - \pi_0)\log\frac{\pi_0}{1-\pi_0} = (x_j - \pi_0)\theta_0,$$

$$\frac{\partial \bar{l}_{\mathrm{W}}^{(\mathrm{e})}}{\partial \theta_0} = \left(\frac{\bar{x} + n^{-1}0.5a}{\pi_0} - \frac{1 - \bar{x} + n^{-1}0.5a}{1-\pi_0} \right) \pi_0(1-\pi_0)$$

$$= \bar{x} - \pi_0 + n^{-1}0.5a(1-2\pi_0) \;\; (= \bar{x} + n^{-1}0.5a - (1+n^{-1}a)\pi_0)$$

$$= \bar{x} - \pi_0 + n^{-1}q_0^*,$$

$$\hat{\pi}_{\mathrm{W}} = \frac{\bar{x} + n^{-1}0.5a}{1+n^{-1}a}, \; \hat{\theta}_{\mathrm{W}} = \log\frac{\hat{\pi}_{\mathrm{W}}}{1-\hat{\pi}_{\mathrm{W}}} = \log\frac{\bar{x} + n^{-1}0.5a}{1 - \bar{x} + n^{-1}0.5a},$$

$$\mathbf{q}_0^* = q_0^* = 0.5a(1-2\pi_0), \quad \frac{\partial q^*}{\partial \theta_0} = -a\bar{i}_0,$$

$$\mathbf{J}_0^{(3)} = j_0^{(3)} = \frac{\partial^3 \bar{l}}{\partial \theta_0^3} = -(1-2\pi_0)\bar{i}_0 = -n^2\kappa_{f3}(\bar{x}) = -n^2\kappa_{f3}\left(\frac{\partial \bar{l}}{\partial \theta_0} \right),$$

$$\mathbf{J}_0^{(4)} = j_0^{(4)} = \frac{\partial^4 \bar{l}}{\partial \theta_0^4} = -(1-6\pi_0+6\pi_0^2)\bar{i}_0 = -n^3\kappa_{f4}(\bar{x}) = -n^3\kappa_{f4}\left(\frac{\partial \bar{l}}{\partial \theta_0} \right),$$

$$\hat{\bar{l}}_{\mathrm{W}}^{(\mathrm{e})} = (\bar{x} + n^{-1}0.5a)\log\hat{\pi}_{\mathrm{W}} + (1 - \bar{x} + n^{-1}0.5a)\log(1-\hat{\pi}_{\mathrm{W}}),$$

$$\hat{\pi}_{\mathrm{W}} = \frac{1}{1+\exp(-\hat{\theta}_{\mathrm{W}})},$$

$$\hat{\bar{l}}_{\mathrm{W}} = \bar{x}\log\hat{\pi}_{\mathrm{W}} + (1-\bar{x})\log(1-\hat{\pi}_{\mathrm{W}}),$$

$$\mathbf{l}_0^{(\mathrm{W})} = l_0^{(\mathrm{W})} = -\lambda^{-1}\frac{\partial q^*}{\partial\theta_0}\left(-\lambda^{-1}\frac{\partial\bar{l}}{\partial\theta_0}\right) - j_0^{(3)}\lambda^{-3}q_0^*\frac{\partial\bar{l}}{\partial\theta_0}$$

$$= \lambda^{-2}(-a\bar{i}_0)(\bar{x}-\pi_0) + (1-2\pi_0)\bar{i}_0(-\bar{i}_0)^{-3}0.5a(1-2\pi_0)(\bar{x}-\pi_0)$$

$$= -a\{\bar{i}_0^{-1} + 0.5\bar{i}_0^{-2}(1-2\pi_0)^2\}(\bar{x}-\pi_0) \ \text{ (see (A1.1))},$$

$$-2\mathrm{E}_f(\hat{\bar{l}}_{\mathrm{W}} - \hat{\bar{l}}_{\mathrm{W}}^*) = 2n^{-1}\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma}) + n^{-2}(c_1+c_2+c_3) + O(n^{-3})$$

$$= n^{-1}b_1 + n^{-2}b_2 + O(n^{-3})$$

$$b_1 = 2\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma}) = -2q = -2,$$

$b_2 = c_1 \ (c_2 = c_3 = 0$ under canonical parametrization; see Corollaries 1 and 2 with (A1.5))

$$= -2(1-2\pi_0)^2\bar{i}_0^{-1} + (1-6\pi_0+6\pi_0^2)\bar{i}_0^{-1}$$

$$-2\frac{\partial q^*}{\partial\theta_0}\lambda^{-2}\gamma + 2j_0^{(3)}\lambda^{-3}\gamma q_0^* + O(n^{-1})$$

$$= -2(1-2\pi_0)^2\bar{i}_0^{-1} + (1-6\pi_0+6\pi_0^2)\bar{i}_0^{-1}$$

$$+2a\bar{i}_0\,\bar{i}_0^{-1} + 2(1-2\pi_0)\bar{i}_0\,\bar{i}_0^{-2}0.5a(1-2\pi_0)$$

$$= -2(1-2\pi_0)^2\bar{i}_0^{-1} + (1-6\pi_0+6\pi_0^2)\bar{i}_0^{-1}$$

$$+a\{(1-2\pi_0)^2\bar{i}_0^{-1} + 2\}$$

$$= -(1-2\pi_0)^2\bar{i}_0^{-1} - 2 + a\{(1-2\pi_0)^2\bar{i}_0^{-1} + 2\}$$

$$= (a-1)\{(1-2\pi_0)^2\bar{i}_0^{-1} + 2\},$$

$$\bar{l}_{\mathrm{ML}}^{(1)} = -2(\bar{l}_0 - \bar{l}_0^*) = -2(\bar{x}-\pi_0)\log\frac{\pi_0}{1-\pi_0} = -2(\bar{x}-\pi_0)\theta_0,$$

$$\bar{l}_{\mathrm{ML}}^{(2)} = \frac{\partial\bar{l}}{\partial\theta_0}\lambda^{-1}\frac{\partial\bar{l}}{\partial\theta_0} = -(\bar{x}-\pi_0)^2\bar{i}_0^{-1},\ n\mathrm{E}_f(\bar{l}_{\mathrm{ML}}^{(2)}) = \lambda^{-1}\gamma = -1,$$

$$\overline{l}_{\mathrm{ML}}^{(3)} = \frac{1}{3} j_0^{(3)} \lambda^{-3} \left(\frac{\partial \overline{l}}{\partial \theta_0} \right)^3 = \frac{1}{3}(1-2\pi_0)\overline{i}_0 \overline{i}_0^{-3} (\overline{x} - \pi_0)^3$$

$$= \frac{1}{3}(1-2\pi_0)\overline{i}_0^{-2} (\overline{x} - \pi_0)^3,$$

$$n^2 \mathrm{E}_f (\overline{l}_{\mathrm{ML}}^{(3)}) = \frac{1}{3}(1-2\pi_0)\overline{i}_0^{-2} n^2 \mathrm{E}_f \{(\overline{x} - \pi_0)^3\} = \frac{1}{3}(1-2\pi_0)^2 \overline{i}_0^{-1}$$

$$= \frac{1}{3}\{\mathrm{sk}(x^*)\}^2,$$

where $\mathrm{sk}(\cdot)$ is the skewness of a variable.

$$\overline{l}_{\mathrm{ML}}^{(4)} = \frac{1}{4}(j_0^{(3)})^2 \lambda^{-5} \left(\frac{\partial \overline{l}}{\partial \theta_0} \right)^4 - \frac{1}{12} j_0^{(4)} \lambda^{-4} \left(\frac{\partial \overline{l}}{\partial \theta_0} \right)^4$$

$$n^2 \mathrm{E}_f (\overline{l}_{\mathrm{ML}}^{(4)}) = -\frac{3}{4}\{\mathrm{sk}(x^*)\}^2 + \frac{1}{4}\mathrm{kt}(x^*) + O(n^{-1})$$

$$= -\frac{1}{4}\{3(1-2\pi_0)^2 - (1-6\pi_0 + 6\pi_0^2)\}\overline{i}_0^{-1} + O(n^{-1})$$

$$= -\frac{1}{4}(6\pi_0^2 - 6\pi_0 + 2)\overline{i}_0^{-1} + O(n^{-1}) = \frac{3}{2} - \frac{1}{2}\overline{i}_0^{-1} + O(n^{-1}),$$

where $\mathrm{kt}(\cdot)$ is the excess kurtosis of a variable,

$$n^2 \mathrm{E}_f (\overline{l}_{\mathrm{ML}}^{(3)} + \overline{l}_{\mathrm{ML}}^{(4)}) = -\frac{5}{12}\{\mathrm{sk}(x^*)\}^2 + \frac{1}{4}\mathrm{kt}(x^*) + O(n^{-1})$$

$$= \left\{ -\frac{5}{12}(1-2\pi_0)^2 + \frac{1}{4}(1-6\pi_0 + 6\pi_0^2) \right\} \overline{i}_0^{-1} + O(n^{-1})$$

$$= \frac{1}{12}(-2 + 2\pi_0 - 2\pi_0^2)\overline{i}_0^{-1} + O(n^{-1}) = \frac{1}{6}(1 - \overline{i}_0^{-1}) + O(n^{-1}),$$

$$-n^{-2}\mathbf{q}_0^{*}{}'\mathbf{\Lambda}^{-1}\mathbf{q}_0^{*} = -n^{-2}(q_0^*)^2 \lambda^{-1} = n^{-2}\frac{a^2}{4}(1-2\pi_0)^2 \overline{i}_0^{-1}.$$

For $t_{\mathrm{W}}^{(\mathrm{A})} = \dfrac{n^{1/2}(n^{-1}\mathrm{AIC}_{\mathrm{W}} + 2\overline{l}_0^*)}{(\hat{v}_{\mathrm{W}}^{(\mathrm{A})})^{1/2}}$,

we have

$$\hat{v}_{\mathrm{W}}^{(\mathrm{A})} = 4(n-1)^{-1}\sum_{j=1}^{n}(\hat{l}_{\mathrm{W}j} - \hat{\overline{l}}_{\mathrm{W}})^2 \equiv \hat{\alpha}_{\mathrm{W}2}^{(\mathrm{A})} = O_p(1),$$

$$\hat{l}_{\mathrm{W}j} = l_j \,|_{\theta=\hat{\theta}_{\mathrm{W}}},\ \hat{\overline{l}}_{\mathrm{W}} = n^{-1}\sum_{j=1}^{n}\hat{l}_{\mathrm{W}j},$$

$$v_0^{(\mathrm{A})} = 4(n-1)^{-1}\sum_{j=1}^{n}(l_{0j} - \overline{l}_0)^2 = 4\theta_0^2(n-1)^{-1}\sum_{j=1}^{n}(x_j - \overline{x})^2$$

$$\equiv 4\theta_0^2 u_x^2,$$

$$\mathrm{E}_f(v_0^{(\mathrm{A})}) = 4\theta_0^2\overline{i}_0 = \alpha_{\mathrm{ML2}}^{(\mathrm{A})}\ \ (\mathrm{E}_f(u_x^2) = \mathrm{var}_f(x^*) = \pi_0(1-\pi_0) = \overline{i}_0\,),$$

$$\mathrm{var}_f(l_{0j}) = \theta_0^2\ \mathrm{var}_f(x_j) = \theta_0^2\overline{i}_0,$$

$$\kappa_{f3}(l_{0j}) = \theta_0^3\kappa_{f3}(x_j) = \theta_0^3(1-2\pi_0)\overline{i}_0,$$

$$\kappa_{f4}(l_{0j}) = \theta_0^4\kappa_{f4}(x_j) = \theta_0^4(1-6\pi_0 + 6\pi_0^2)\overline{i}_0,$$

$$\begin{aligned}\mathrm{E}_f\{(l_{0j} - \overline{l}_0^*)^4\} &= \theta_0^4[\kappa_{f4}(x_j) + 3\{\mathrm{var}_f(x_j)\}^2] \\ &= \theta_0^4\{1-6\pi_0 + 6\pi_0^2 + 3\pi_0(1-\pi_0)\}\overline{i}_0 \\ &= \theta_0^4(1-3\pi_0 + 3\pi_0^2)\overline{i}_0,\end{aligned}$$

$$\mathrm{E}_f\left(l_{0j}\frac{\partial l_j}{\partial\theta_0}\right) = \theta_0\ \mathrm{var}_f(x_j) = \theta_0\overline{i}_0, \qquad \text{(S7.2)}$$

$$\mathrm{E}_f\left\{\frac{\partial l_j}{\partial\theta_0}(l_{0j}-\bar{l}_0^*)^2\right\}=\theta_0^2\kappa_{f3}(x_j)=\theta_0^2(1-2\pi_0)\bar{i}_0,$$

$$\mathrm{E}_f\left\{(l_{0j}-\bar{l}_0^*)\left(\frac{\partial l_j}{\partial\theta_0}\right)^2\right\}=\theta_0\kappa_{f3}(x_j)=\theta_0(1-2\pi_0)\bar{i}_0,$$

$$\mathrm{E}_f\left(\frac{\partial v_0^{(\mathrm{A})}}{\partial\theta_0}\right)=8\mathrm{E}_f\left(l_{0j}\frac{\partial l_j}{\partial\theta_0}\right)=8\theta_0\,\mathrm{var}_f(x_j)=8\theta_0\bar{i}_0,$$

$$\mathrm{E}_f\left(\frac{\partial^2 v_0^{(\mathrm{A})}}{\partial\theta_0^2}\right)=8\gamma=8\bar{i}_0$$

(8γ is due to canonical parametrization),

$$\mathrm{E}_f\left\{\left(\frac{\partial v_0^{(\mathrm{A})}}{\partial\theta_0}\right)^2\right\}=64\left\{\mathrm{E}_f\left(l_{0j}\frac{\partial l_j}{\partial\theta_0}\right)\right\}^2+O(n^{-1})=64\theta_0^2\bar{i}_0^2+O(n^{-1}),$$

$$n\,\mathrm{cov}_f(m_v,\bar{l}_0)=n\,\mathrm{cov}_f(v_0^{(\mathrm{A})},\bar{l}_0)=4\kappa_{f3}(l_{0j})$$

$$=4\theta_0^3\kappa_{f3}(x_j)=4\theta_0^3(1-2\pi_0)\bar{i}_0,$$

$$n\,\mathrm{cov}_f\left(\frac{\partial\bar{l}_0}{\partial\theta_0},m_v\right)=4\mathrm{E}_f\left\{(l_{0j}-\bar{l}_0^*)^2\frac{\partial l_j}{\partial\theta_0}\right\}=4\theta_0^2(1-2\pi_0)\bar{i}_0,$$

$$n\,\mathrm{avar}_f(m_v)=16\left[\mathrm{E}_f\{(l_{0j}-\bar{l}_0^*)^4\}-[\mathrm{E}_f\{(l_{0j}-\bar{l}_0^*)^2\}]^2\right]$$

$$=16\{\theta_0^4(1-3\pi_0+3\pi_0^2)\bar{i}_0-\theta_0^4\bar{i}_0^2\}$$

$$=16\theta_0^4\{1-3\pi_0+3\pi_0^2-\pi_0(1-\pi_0)\}\bar{i}_0=16\theta_0^4(1-2\pi_0)^2\bar{i}_0,$$

$$n\,\mathrm{cov}_f\left(\frac{\partial v_0^{(\mathrm{A})}}{\partial\theta_0},\frac{\partial\bar{l}}{\partial\theta_0}\right)=8\mathrm{E}_f\left\{(l_{0j}-\bar{l}_0^*)\left(\frac{\partial l_j}{\partial\theta_0}\right)^2\right\}=8\theta_0\kappa_{f3}(x_j)$$

$$=8\theta_0(1-2\pi_0)\bar{i}_0$$

$$n\,\mathrm{cov}_f\left(\frac{\partial v_0^{(\mathrm{A})}}{\partial\theta_0},\,\bar{l}_0\right)=8\mathrm{E}_f\left\{(l_{0j}-\bar{l}_0^*)^2\,\frac{\partial l_j}{\partial\theta_0}\right\}$$

$$=8\theta_0^2\kappa_{f3}(x_j)=8\theta_0^2(1-2\pi_0)\bar{i}_0,$$

$$n^2\mathrm{E}_f\{(\bar{l}_{\mathrm{ML}}^{(1)})^2\,\mathbf{m}_v^{(1)}\,'\}=\left[16\kappa_{f4}(l_{0j}),4\mathrm{E}_f\left\{(l_{0j}-\bar{l}_0^*)^2\,\frac{\partial l_j}{\partial\theta_0}\right\}\right]$$

$$=\{16\theta_0^4(1-6\pi_0+6\pi_0^2)\bar{i}_0,\,4\theta_0^2(1-2\pi_0)\bar{i}_0\}$$

(see the result of the second term for (S1.7)),

$$\mathbf{m}_v^{(1)}=\left(m_v,\frac{\partial\bar{l}}{\partial\theta_0}\right)'=\{4\theta_0^2(u_x^2-\bar{i}_0),\,\bar{x}-\pi_0\}',$$

$$\mathbf{m}_v^{(2)}=\left[m_v^2,\;m_v\frac{\partial\bar{l}}{\partial\theta_0},\;0,\;\left(\frac{\partial\bar{l}}{\partial\theta_0}\right)^2,\;\left\{\frac{\partial v^{(\mathrm{A})}}{\partial\theta_0}-\mathrm{E}_f\left(\frac{\partial v^{(\mathrm{A})}}{\partial\theta_0}\right)\right\}\otimes\frac{\partial\bar{l}}{\partial\theta_0}\right]'$$

$$=\{16\theta_0^4(u_x^2-\bar{i}_0)^2,\;4\theta_0^2(u_x^2-\bar{i}_0)(\bar{x}-\pi_0),\,0,\,(\bar{x}-\pi_0)^2,$$

$$8\theta_0(u_x^2-\bar{i}_0)(\bar{x}-\pi_0)\}',$$

$$\mathbf{v}^{(1)}=\frac{1}{2}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}\left\{-1,\;\mathrm{E}_f\left(\frac{\partial v^{(\mathrm{A})}}{\partial\theta_0}\right)\lambda^{-1}\right\}'$$

$$=\frac{1}{2}(4\theta_0^2\bar{i}_0)^{-3/2}\{-1,\,8\theta_0\bar{i}_0(-\bar{i}_0)^{-1}\}'=-\frac{1}{16}(\theta_0^2\bar{i}_0)^{-3/2}(1,\;8\theta_0)',$$

$$\mathbf{v}^{(2)} = \Big[_{\rm (A)} \ \frac{3}{8}(\alpha_{\rm ML2}^{\rm (A)})^{-5/2},\ -\frac{3}{4}(\alpha_{\rm ML2}^{\rm (A)})^{-5/2}\,\mathrm{E}_f\left(\frac{\partial v^{\rm (A)}}{\partial \theta_0}\right)\lambda^{-1},$$

$$-\frac{1}{2}(\alpha_{\rm ML2}^{\rm (A)})^{-3/2}\,\mathrm{E}_f\left(\frac{\partial v^{\rm (A)}}{\partial \theta_0}\right)(\lambda^{-1})^2,\ \Big[_{\rm (B)} \ \frac{1}{4}(\alpha_{\rm ML2}^{\rm (A)})^{-3/2}\,\mathrm{E}_f\left(\frac{\partial v^{\rm (A)}}{\partial \theta_0}\right)\lambda^{-1}\dot{j}_0^{(3)}$$

$$-\frac{1}{4}(\alpha_{\rm ML2}^{\rm (A)})^{-3/2}\,\mathrm{E}_g\left(\frac{\partial^2 v^{\rm (A)}}{\partial \theta_0^2}\right)+\frac{3}{8}(\alpha_{\rm ML2}^{\rm (A)})^{-5/2}\,\mathrm{E}_g\left\{\left(\frac{\partial v^{\rm (A)}}{\partial \theta_0}\right)^2\right\}$$

$$+O(n^{-1}) \ \Big]_{\rm (B)}(\lambda^{-1})^2,\ \frac{1}{2}(\alpha_{\rm ML2}^{\rm (A)})^{-3/2}\lambda^{-1} \ \Big]_{\rm (A)}'$$

$$=\Big[_{\rm (A)} \ \frac{3}{8}(4\theta_0^2\bar{i}_0)^{-5/2}, -\frac{3}{4}(4\theta_0^2\bar{i}_0)^{-5/2}8\theta_0\bar{i}_0(-\bar{i}_0)^{-1}, -\frac{1}{2}(4\theta_0^2\bar{i}_0)^{-3/2}8\theta_0\bar{i}_0\bar{i}_0^{-2},$$

$$\Big[_{\rm (B)} \ \frac{1}{4}(4\theta_0^2\bar{i}_0)^{-3/2}8\theta_0\bar{i}_0(-\bar{i}_0)^{-1}\{-(1-2\pi_0)\bar{i}_0\} - \frac{1}{4}(4\theta_0^2\bar{i}_0)^{-3/2}8\bar{i}_0$$

$$+\frac{3}{8}(4\theta_0^2\bar{i}_0)^{-5/2}64\theta_0^2\bar{i}_0^2 \ \Big]_{\rm (B)}\bar{i}_0^{-2}, \frac{1}{2}(4\theta_0^2\bar{i}_0)^{-3/2}(-\bar{i}_0)^{-1} \ \Big]_{\rm (A)}'$$

$$=\Big[\frac{3}{256}\theta_0^{-5}\bar{i}_0^{-5/2}, \frac{3}{16}\theta_0^{-4}\bar{i}_0^{-5/2}, -\frac{1}{2}\theta_0^{-2}\bar{i}_0^{-5/2},$$

$$\left\{\frac{1}{4}\theta_0^{-2}\bar{i}_0^{-1/2}(1-2\pi_0) - \frac{1}{4}\theta_0^{-3}\bar{i}_0^{-1/2} + \frac{3}{4}\theta_0^{-3}\bar{i}_0^{-1/2}\right\}\bar{i}_0^{-2},\ -\frac{1}{16}\theta_0^{-3}\bar{i}_0^{-5/2}\Big]$$

$$=\Big[\frac{3}{256}\theta_0^{-5}, \frac{3}{16}\theta_0^{-4}, -\frac{1}{2}\theta_0^{-2}, \frac{1}{4}\theta_0^{-2}(1-2\pi_0)+\frac{1}{2}\theta_0^{-3}, -\frac{1}{16}\theta_0^{-3}\Big]'\bar{i}_0^{-5/2}$$

$$\equiv \mathbf{v}^{(2)*}\bar{i}_0^{-5/2},$$

$$n\mathrm{E}_f(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)} = -2\left\{n\,\mathrm{cov}_f(\bar{l}_0, m_v),\ n\,\mathrm{cov}_f\left(\bar{l}_0, \frac{\partial \bar{l}}{\partial \theta_0}\right)\right\}\mathbf{v}^{(1)}$$

$$= -2\left\{4\kappa_{f3}(l_{0j}), n\,\mathrm{cov}_f\left(\bar{l}_0, \frac{\partial \bar{l}}{\partial \theta_0}\right)\right\}\mathbf{v}^{(1)}$$

$$= -\{8\theta_0^3(1-2\pi_0)\bar{i}_0,\ 2\theta_0\bar{i}_0\}\mathbf{v}^{(1)}$$

$$= \{8\theta_0^3(1-2\pi_0)\bar{i}_0,\ 2\theta_0\bar{i}_0\}(1,\ 8\theta_0)'\frac{1}{16}(\theta_0^2\bar{i}_0)^{-3/2}$$

$$= \frac{1}{2}(1-2\pi_0)\bar{i}_0^{-1/2} + \theta_0^{-1}\bar{i}_0^{-1/2} = \left\{\frac{1}{2}(1-2\pi_0) + \theta_0^{-1}\right\}\bar{i}_0^{-1/2},$$

$$(\hat{v}_{\mathrm{W}}^{(\mathrm{A})})^{-1/2} = (\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1/2} + n^{-1}\eta_{\mathrm{W}}^{(v)} + \sum_{j=1}^{2}\mathbf{v}^{(j)}{}'\mathbf{m}_v^{(j)} + O_p(n^{-3/2}),$$

$$\eta_{\mathrm{W}}^{(v)} = \frac{1}{2}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-3/2}\mathrm{E}_f\left(\frac{\partial v^{(\mathrm{A})}}{\partial \theta_0}\right)\lambda^{-1}q_0^* \quad \text{(see (S1.2))}$$

$$= \frac{1}{2}(4\theta_0^2\bar{i}_0)^{-3/2}8\theta_0\bar{i}_0(-\bar{i}_0)^{-1}0.5a(1-2\pi_0)$$

$$= -\frac{a}{4}\theta_0^{-2}\bar{i}_0^{-3/2}(1-2\pi_0).$$

S7.2 $n^{-1}\mathrm{AIC}_{\mathrm{W}}$

$$n^{-1}\mathrm{AIC}_{\mathrm{W}} = -2\hat{\bar{l}}_{\mathrm{W}} + n^{-1}2q = -2\hat{\bar{l}}_{\mathrm{W}} + n^{-1}2,$$

where note that $\hat{\bar{l}}_{\mathrm{W}}$ is used rather than $\hat{\bar{l}}_{\mathrm{W}}^{(\mathrm{e})}$.

S7.2.1 Asymptotic cumulants of $n^{-1}\mathrm{AIC}_{\mathrm{ML}}$ before studentization

For estimation of $-2\mathrm{E}_f(\hat{\bar{l}}_{\mathrm{W}}^*)$,

$$
\begin{aligned}
&\kappa_{f1}\{n^{-1}\mathrm{AIC}_{\mathrm{W}} + 2\mathrm{E}_f(\hat{\bar{l}}_{\mathrm{W}}^{*})\} \\
&\quad = n^{-1}(2\lambda^{-1}\gamma + 2q) + n^{-2}b_2 + O(n^{-3}) \\
&\quad = n^{-1} \times 0 + n^{-2}b_2 + O(n^{-3}) = n^{-2}c_1 + O(n^{-3}) \\
&\quad = n^{-2}(a-1)\{(1-2\pi_0)^2\bar{i}_0^{-1} + 2\} + O(n^{-3})
\end{aligned}
\qquad \text{(S7.3)}
$$

($b_2 = c_1$ due to canonical parametrization; $\alpha_{\mathrm{W1}}^{(\mathrm{A})*} = \alpha_{\mathrm{ML1}}^{(\mathrm{A})*} = b_1 = 0$, $\alpha_{\mathrm{W\Delta 1}}^{(\mathrm{A})*} = b_2 = c_1$),

while for estimation of $-2\bar{l}_0^{*}$,

$$
\begin{aligned}
&\kappa_{f1}(n^{-1}\mathrm{AIC}_{\mathrm{W}} + 2\bar{l}_0^{*}) \\
&\quad = n^{-1}(\lambda^{-1}\gamma + 2q) + n^{-2}\{n^2\mathrm{E}_f(\bar{l}_{\mathrm{ML}}^{(3)} + \bar{l}_{\mathrm{ML}}^{(4)}) + (q_0^{*})^2\bar{i}_0^{-1}\} + O(n^{-3}) \\
&\quad = n^{-1} + n^{-2}\left\{\frac{1}{6}(1-\bar{i}_0^{-1}) + \frac{a^2}{4}(1-2\pi_0)^2\bar{i}_0^{-1}\right\} + O(n^{-3}) \qquad \text{(S7.4)} \\
&= n^{-1}\alpha_{\mathrm{ML1}}^{(\mathrm{A})} + n^{-2}\alpha_{\mathrm{W\Delta 1}}^{(\mathrm{A})} + O(n^{-3}) \quad (\alpha_{\mathrm{W1}}^{(\mathrm{A})} = \alpha_{\mathrm{ML1}}^{(\mathrm{A})}).
\end{aligned}
$$

$$
\begin{aligned}
&\kappa_{f2}(n^{-1}\mathrm{AIC}_{\mathrm{W}}) = n^{-1}[n\mathrm{E}_f\{(\bar{l}_{\mathrm{ML}}^{(1)})^2\}] + n^{-2}[2n^2\mathrm{E}_f(\bar{l}_{\mathrm{ML}}^{(1)}\bar{l}_{\mathrm{ML}}^{(2)}) \\
&\qquad + 2n^2\mathrm{E}_f(\bar{l}_{\mathrm{ML}}^{(1)}\bar{l}_{\mathrm{ML}}^{(3)}) + n^2\mathrm{E}_f\{(\bar{l}_{\mathrm{ML}}^{(2)})^2\} - \{n\mathrm{E}_f(\bar{l}_{\mathrm{ML}}^{(2)})\}^2] + O(n^{-3}) \\
&= n^{-1}4n\,\mathrm{var}_f(\bar{x})\theta_0^2 + n^{-2}\Bigl[\; 4n^2\kappa_{f3}(\bar{x})\theta_0\bar{i}_0^{-1} \qquad \text{(S7.5)} \\
&\qquad -\frac{4}{3}(1-2\pi_0)\theta_0\bar{i}_0^{-2}n^2\mathrm{E}_f\{(\bar{x}-\pi_0)^4\} + 2\lambda^{-2}\gamma^2 \Bigr] + O(n^{-3}) \\
&= n^{-1}4\theta_0^2\bar{i}_0 + n^{-2}\{4(1-2\pi_0)\theta_0 - 4(1-2\pi_0)\theta_0 + 2\} + O(n^{-3}) \\
&= n^{-1}4\theta_0^2\bar{i}_0 + n^{-2}2 + O(n^{-3}) = n^{-1}\alpha_{\mathrm{ML2}}^{(\mathrm{A})} + n^{-2}\alpha_{\mathrm{ML\Delta 2}}^{(\mathrm{A})} + O(n^{-3}) \\
&(\alpha_{\mathrm{W2}}^{(\mathrm{A})} = \alpha_{\mathrm{ML2}}^{(\mathrm{A})},\, \alpha_{\mathrm{W\Delta 2}}^{(\mathrm{A})} = \alpha_{\mathrm{ML\Delta 2}}^{(\mathrm{A})}).
\end{aligned}
$$

$$
\begin{aligned}
\kappa_{f3}(n^{-1}\mathrm{AIC}_{\mathrm{W}}) &= n^{-2}[\, n^2\mathrm{E}_f\{(\bar{l}_{\mathrm{ML}}^{(1)})^3\} + 3n^2\mathrm{E}_f\{(\bar{l}_{\mathrm{ML}}^{(1)})^2\bar{l}_{\mathrm{ML}}^{(2)})\} \\
&\qquad - 3n\mathrm{E}_f(\bar{l}_{\mathrm{ML}}^{(2)})\alpha_{\mathrm{ML2}}^{(\mathrm{A})}\ ] + O(n^{-3}) \\
&= n^{-2}\left[n^2\mathrm{E}_f\{(\bar{l}_{\mathrm{ML}}^{(1)})^3\} + 6\lambda^{-1}\left\{ n\,\mathrm{cov}_f\left(\bar{l}_{\mathrm{ML}}^{(1)}, \frac{\partial \bar{l}}{\partial \theta_0} \right) \right\}^2 \right] + O(n^{-3}) \\
&= n^{-2}\left[n^2\mathrm{E}_f\{(\bar{l}_{\mathrm{ML}}^{(1)})^3\} + 24\lambda^{-1}\theta_0^2\left\{ n\,\mathrm{cov}_f\left(\bar{x}, \frac{\partial \bar{l}}{\partial \theta_0} \right) \right\}^2 \right] + O(n^{-3}) \\
&= n^{-2}\{-8\theta_0^3 n^2\kappa_{f3}(\bar{x}) - 24\bar{i}_0^{\,-1}\theta_0^2\bar{i}_0^{\,2}\} + O(n^{-3}) \\
&= n^{-2}\{-8\theta_0^3(1-2\pi_0) - 24\theta_0^2\}\bar{i}_0 + O(n^{-3}) \\
&= n^{-2}\alpha_{\mathrm{ML3}}^{(\mathrm{A})} + O(n^{-3})\ \ (\alpha_{\mathrm{W3}}^{(\mathrm{A})} = \alpha_{\mathrm{ML3}}^{(\mathrm{A})}).
\end{aligned}
\tag{S7.6}
$$

$$
\begin{aligned}
\kappa_{f4}(n^{-1}\mathrm{AIC}_{\mathrm{W}}) &= n^{-3}\ \underset{(\mathrm{A})}{[}\ n^3\kappa_{f4}(\bar{l}_{\mathrm{ML}}^{(1)}) + 4n^3\mathrm{E}_f\{(\bar{l}_{\mathrm{ML}}^{(1)})^3\bar{l}_{\mathrm{ML}}^{(2)}\} \\
&\quad + 6n^3\mathrm{E}_f\{(\bar{l}_{\mathrm{ML}}^{(1)})^2(\bar{l}_{\mathrm{ML}}^{(2)})^2\} + 4n^3\mathrm{E}_f\{(\bar{l}_{\mathrm{ML}}^{(1)})^3\bar{l}_{\mathrm{ML}}^{(3)}\} - 4n\mathrm{E}_f(\bar{l}_{\mathrm{ML}}^{(2)})\alpha_{\mathrm{ML3}}^{(\mathrm{A})} \\
&\quad - 6\alpha_{\mathrm{ML2}}^{(\mathrm{A})}\alpha_{\mathrm{ML\Delta 2}}^{(\mathrm{A})} - 6\alpha_{\mathrm{ML2}}^{(\mathrm{A})}\{n\mathrm{E}_f(\bar{l}_{\mathrm{ML}}^{(2)})\}^2\ \underset{(\mathrm{A})}{]} + O(n^{-4}) \\
&= n^{-3}\Bigg[\ 16\theta_0^4 n^3\kappa_{f4}(\bar{x}) - 32\lambda^{-1}\theta_0^3 n^3\mathrm{E}_f\{(\bar{x}-\pi_0)^5\} \\
&\quad + 24\lambda^{-2}\theta_0^2 n^3\mathrm{E}_f\{(\bar{x}-\pi_0)^6\} + 4\times(-8)\frac{1}{3}\theta_0^3(1-2\pi_0)\bar{i}_0^{\,-2} n^3\mathrm{E}_f\{(\bar{x}-\pi_0)^6\} \\
&\quad - 4\times(-1)\alpha_{\mathrm{ML3}}^{(\mathrm{A})} - 6\alpha_{\mathrm{ML2}}^{(\mathrm{A})}\alpha_{\mathrm{ML\Delta 2}}^{(\mathrm{A})} - 6\alpha_{\mathrm{ML2}}^{(\mathrm{A})}(-1)^2\ \Bigg] + O(n^{-4})
\end{aligned}
$$

$$
\begin{aligned}
&= n^{-3}\Bigl[\ 16\theta_0^4(1-6\pi_0+6\pi_0^2)\bar{i}_0 + 32\theta_0^3\bar{i}_0^{-1}10\bar{i}_0(1-2\pi_0)\bar{i}_0 \\
&\qquad + 24\times 15\theta_0^2\bar{i}_0^{-2}\bar{i}_0^3 - \frac{32}{3}\times 15\theta_0^3(1-2\pi_0)\bar{i}_0^{-2}\bar{i}_0^3 \\
&\qquad - 4\{8\theta_0^3(1-2\pi_0)+24\theta_0^2\}\bar{i}_0 - 6\times 4\theta_0^2\bar{i}_0\times 2 - 6\times 4\theta_0^2\bar{i}_0\ \Bigr] \qquad \text{(S7.7)} \\
&\quad + O(n^{-4}) \\
&= n^{-3}\{\ 16\theta_0^4(1-6\pi_0+6\pi_0^2) + (320-160-32)\theta_0^3(1-2\pi_0) \\
&\qquad + (360-96-48-24)\theta_0^2\ \}\bar{i}_0 + O(n^{-4}) \\
&= n^{-3}\{\ 16\theta_0^4(1-6\pi_0+6\pi_0^2) + 128\theta_0^3(1-2\pi_0) + 192\theta_0^2\ \}\bar{i}_0 + O(n^{-4}) \\
&= n^{-3}\alpha_{\mathrm{ML4}}^{(\mathrm{A})} + O(n^{-4})\ \ (\alpha_{\mathrm{W4}}^{(\mathrm{A})} = \alpha_{\mathrm{ML4}}^{(\mathrm{A})}).
\end{aligned}
$$

S7.2.2 Asymptotic cumulants of $n^{-1}\mathrm{AIC}_{\mathrm{W}}$ after studentization for estimation of $-2\bar{l}_0^*$

$$
t_{\mathrm{W}}^{(\mathrm{A})} = \frac{n^{1/2}(n^{-1}\mathrm{AIC}_{\mathrm{W}} + 2\bar{l}_0^*)}{(\hat{v}_{\mathrm{W}}^{(\mathrm{A})})^{1/2}}.
$$

$$
\begin{aligned}
\kappa_{f1}(t_{\mathrm{W}}^{(\mathrm{A})}) &= n^{-1/2}\{\alpha_{\mathrm{ML1}}^{(\mathrm{A})}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} + \alpha_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})}\} + O(n^{-3/2}) \\
&= n^{-1/2}\{\alpha_{\mathrm{ML1}}^{(\mathrm{A})}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} + n\mathrm{E}_f(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)}\} + O(n^{-3/2}) \\
&= n^{-1/2}\left[1\times(4\theta_0^2\bar{i}_0)^{-1/2} + \left\{\frac{1}{2}(1-2\pi_0) + \theta_0^{-1}\right\}\bar{i}_0^{-1/2}\right] + O(n^{-3/2}) \\
&= n^{-1/2}\left\{\frac{3}{2}\theta_0^{-1} + \frac{1}{2}(1-2\pi_0)\right\}\bar{i}_0^{-1/2} + O(n^{-3/2}) = n^{-1/2}\alpha_{(t)\mathrm{ML1}}^{(\mathrm{A})} + O(n^{-3/2}) \qquad \text{(S7.8)}
\end{aligned}
$$

$$
\begin{aligned}
&(\alpha_{(t)\mathrm{W1}}^{(\mathrm{A})} = \alpha_{(t)\mathrm{ML1}}^{(\mathrm{A})}; \\
&\ \alpha_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})} = n\mathrm{E}_f(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)} = \{(1/2)(1-2\pi_0) + \theta_0^{-1}\}\bar{i}_0^{-1/2}).
\end{aligned}
$$

$$\kappa_{f2}(t_{\mathrm{W}}^{(\mathrm{A})})$$

$$= 1 + n^{-1}\underset{(\mathrm{A})}{\Big[}\ \alpha^{(\mathrm{A})}_{\mathrm{ML}\Delta 2}(\alpha^{(\mathrm{A})}_{\mathrm{ML}2})^{-1} + 2n^2 \mathrm{E}_f\{(\overline{l}^{(1)}_{\mathrm{ML}})^2 \mathbf{v}^{(1)}{}'\mathbf{m}^{(1)}_v\}(\alpha^{(\mathrm{A})}_{\mathrm{ML}2})^{-1/2}$$

$$+ 2n^2 \mathrm{E}_f \underset{(\mathrm{B})}{\Big[}\ \overline{l}^{(1)}_{\mathrm{ML}}\{\overline{l}^{(2)}_{\mathrm{ML}}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)} + \overline{l}^{(1)}_{\mathrm{ML}}(n^{-1}\eta_{\mathrm{W}}^{(v)} + \mathbf{v}^{(2)}{}'\mathbf{m}_v^{(2)}) + n^{-1}2q\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}\}\ \underset{(\mathrm{B})}{\Big]}$$

$$\times(\alpha^{(\mathrm{A})}_{\mathrm{ML}2})^{-1/2}$$

$$+ n^2\mathrm{E}_f\{2\overline{l}^{(2)}_{\mathrm{ML}}(\alpha^{(\mathrm{A})}_{\mathrm{ML}2})^{-1/2}\overline{l}^{(1)}_{\mathrm{ML}}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)} + (\overline{l}^{(1)}_{\mathrm{ML}}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)})^2\}$$

$$-\{2n\mathrm{E}_f(\overline{l}^{(2)}_{\mathrm{ML}})(\alpha^{(\mathrm{A})}_{\mathrm{ML}2})^{-1/2}\alpha^{(\mathrm{A})}_{(\Delta t)\mathrm{ML}1} + (\alpha^{(\mathrm{A})}_{(\Delta t)\mathrm{ML}1})^2\}\ \underset{(\mathrm{A})}{\Big]} + O(n^{-2}),$$

(S7.9)

(i) the first term in $\underset{(\mathrm{A})}{[}\ \cdot\ \underset{(\mathrm{A})}{]}$ of (S7.9) is

$$\alpha^{(\mathrm{A})}_{\mathrm{ML}\Delta 2}(\alpha^{(\mathrm{A})}_{\mathrm{ML}2})^{-1} = 2(4\theta_0^2\overline{i}_0)^{-1} = \frac{1}{2}\theta_0^{-2}\overline{i}_0^{-1},$$

(ii) the second term in $\underset{(\mathrm{A})}{[}\ \cdot\ \underset{(\mathrm{A})}{]}$ of (S7.9) is

$$2n^2\mathrm{E}_f\{(\overline{l}^{(1)}_{\mathrm{ML}})^2\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}\}(\alpha^{(\mathrm{A})}_{\mathrm{ML}2})^{-1/2}$$

$$= 2\left[16\kappa_{f4}(l_{0j}), 4\mathrm{E}_f\left\{(l_{0j}-\overline{l}_0^*)^2\frac{\partial l_j}{\partial\theta_0}\right\}\right]\mathbf{v}^{(1)}(\alpha^{(\mathrm{A})}_{\mathrm{ML}2})^{-1/2}\quad (\mathrm{see\,(S1.7)})$$

$$= 2\{16\theta_0^4(1-6\pi_0+6\pi_0^2)\overline{i}_0,\ 4\theta_0^2(1-2\pi_0)\overline{i}_0\}$$

$$\times\left(-\frac{1}{16}\right)(\theta_0^2\overline{i}_0)^{-3/2}(1,\ 8\theta_0)'(4\theta_0^2\overline{i}_0)^{-1/2}$$

$$= -\{\theta_0^4(1-6\pi_0+6\pi_0^2)\overline{i}_0 + 2\theta_0^3(1-2\pi_0)\overline{i}_0\}(\theta_0^2\overline{i}_0)^{-2}$$

$$= -\{1-6\pi_0+6\pi_0^2+2\theta_0^{-1}(1-2\pi_0)\}\overline{i}_0^{-1}$$

$$= -\{(1-2\pi_0)^2-2\overline{i}_0+2\theta_0^{-1}(1-2\pi_0)\}\overline{i}_0^{-1}$$

$$= -\{(1-2\pi_0)^2+2\theta_0^{-1}(1-2\pi_0)\}\overline{i}_0^{-1}+2,$$

(iii) the first part of $2n^2\mathrm{E}_f \underset{\text{(B)}}{[} \cdot \underset{\text{(B)}}{]} (\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}$ in (S7.9) is

$$2n^2\mathrm{E}_f(\overline{l}_{\mathrm{ML}}^{(1)}\overline{l}_{\mathrm{ML}}^{(2)}\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}$$

$$= -4\left[\lambda^{-1}\gamma\left\{n\,\mathrm{cov}_f(\overline{l}_0, m_v),\ \mathrm{E}_f\left(l_{0j}\frac{\partial l_j}{\partial\theta_0}\right)\right\}\right.$$

$$\left.+2\mathrm{E}_f\left(l_{0j}\frac{\partial l_j}{\partial\theta_0}\right)\lambda^{-1}\left\{n\,\mathrm{cov}_f\left(\frac{\partial\overline{l}}{\partial\theta_0}, m_v\right), \gamma\right\}\right]\mathbf{v}^{(1)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}$$

$$+O(n^{-1})$$

$$= -4[\ -\{4\theta_0^3(1-2\pi_0)\overline{i}_0,\ \theta_0\overline{i}_0\} + 2\theta_0\overline{i}_0(-\overline{i}_0)^{-1}\{4\theta_0^2(1-2\pi_0)\overline{i}_0,\ \overline{i}_0\}\ ]$$

$$\times\left(-\frac{1}{16}\right)(\theta_0^2\overline{i}_0)^{-3/2}(1,\ 8\theta_0)'(4\theta_0^2\overline{i}_0)^{-1/2} + O(n^{-1})$$

$$= -\frac{1}{8}[\ \{4\theta_0^3(1-2\pi_0),\ \theta_0\}\overline{i}_0 + 2\theta_0\{4\theta_0^2(1-2\pi_0),\ 1\}\overline{i}_0\ \ ]$$

$$\times(1,\ 8\theta_0)'(\theta_0^2\overline{i}_0)^{-2} + O(n^{-1})$$

$$= -\frac{1}{8}[\ 4\theta_0^3(1-2\pi_0) + 8\theta_0^2 + 2\theta_0\{4\theta_0^2(1-2\pi_0) + 8\theta_0\}\ \ ]\theta_0^{-4}\overline{i}_0^{-1} + O(n^{-1})$$

$$= -\left\{\frac{3}{2}\theta_0^{-1}(1-2\pi_0) + 3\theta_0^{-2}\right\}\overline{i}_0^{-1} + O(n^{-1}),$$

(iv) the central part of $2n^2\mathrm{E}_f \underset{\text{(B)}}{[} \cdot \underset{\text{(B)}}{]} (\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}$ in (S7.9) is

$$2n^2\mathrm{E}_f\{(\overline{l}_{\mathrm{ML}}^{(1)})^2(n^{-1}\eta_{\mathrm{W}}^{(v)} + \mathbf{v}^{(2)}{}'\mathbf{m}_v^{(2)})\}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}$$

$$= 2\alpha_{\mathrm{ML2}}^{(\mathrm{A})}\eta_{\mathrm{W}}^{(v)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} + 2\alpha_{\mathrm{ML2}}^{(\mathrm{A})}\mathrm{E}_f(\mathbf{m}_v^{(2)}{}')\mathbf{v}^{(2)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}$$

$$+2\times(-2)^2\left[2\{n\,\mathrm{cov}_f(\overline{l}_0, m_v)\}^2,\ 2n\,\mathrm{cov}_f(\overline{l}_0, m_v)\mathrm{E}_f\left(l_{0j}\frac{\partial l_j}{\partial\theta_0}\right), 0,\right.$$

$$\left.2\left\{\mathrm{E}_f\left(l_{0j}\frac{\partial l_j}{\partial\theta_0}\right)\right\}^2, 2n\,\mathrm{cov}_f\left(\overline{l}_0, \frac{\partial v^{(\mathrm{A})}}{\partial\theta_0}\right)\mathrm{E}_f\left(l_{0j}\frac{\partial l_j}{\partial\theta_0}\right)\right]\mathbf{v}^{(2)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}$$

$$+O(n^{-1})$$

$$
\begin{aligned}
&= 2(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{1/2}\eta_{\mathrm{W}}^{(v)} + 2(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{1/2} \\
&\quad \times \left[n\,\mathrm{avar}_f(m_v),\, n\,\mathrm{cov}_f\left(m_v, \frac{\partial \bar{l}}{\partial \theta_0} \right), 0, \gamma, n\,\mathrm{cov}_f\left(\frac{\partial v^{(\mathrm{A})}}{\partial \theta_0}, \frac{\partial \bar{l}}{\partial \theta_0} \right) \right] \mathbf{v}^{(2)} \\
&\quad + 8[\ 2\{4\theta_0^3(1-2\pi_0)\bar{i}_0\}^2,\ 8\theta_0^3(1-2\pi_0)\bar{i}_0\theta_0\bar{i}_0,\ 0,\ 2(\theta_0\bar{i}_0)^2, \\
&\qquad 16\theta_0^2(1-2\pi_0)\bar{i}_0\theta_0\bar{i}_0\ \ ]\mathbf{v}^{(2)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} + O(n^{-1}), \\
&= 2(4\theta_0^2\bar{i}_0)^{1/2}\left\{ -\frac{a}{4}\theta_0^{-2}\bar{i}_0^{-3/2}(1-2\pi_0) \right\} \\
&\quad + 2[\ 16\theta_0^4(1-2\pi_0)^2\bar{i}_0,\ 4\theta_0^2(1-2\pi_0)\bar{i}_0,\ 0,\ \bar{i}_0,\ 8\theta_0(1-2\pi_0)\bar{i}_0\ \ ] \\
&\quad \times \mathbf{v}^{(2)*}\bar{i}_0^{-5/2}(4\theta_0^2\bar{i}_0)^{1/2} \\
&\quad + 8[\ \ 32\theta_0^6(1-2\pi_0)^2\bar{i}_0^{\,2},\ 8\theta_0^4(1-2\pi_0)\bar{i}_0^{\,2},\ 0,\ 2\theta_0^2\bar{i}_0^{\,2}, \\
&\qquad 16\theta_0^3(1-2\pi_0)\bar{i}_0^{\,2}\ \ ]\mathbf{v}^{(2)*}\bar{i}_0^{-5/2}(4\theta_0^2\bar{i}_0)^{-1/2} + O(n^{-1}) \\
&= -\frac{a}{4}\theta_0^{-1}\bar{i}_0^{-1}(1-2\pi_0) + [\ \ 64\theta_0^5(1-2\pi_0)^2\bar{i}_0^{-1},\ 16\theta_0^3(1-2\pi_0)\bar{i}_0^{-1}, \\
&\qquad 0,\ 4\theta_0\bar{i}_0^{-1},\ 32\theta_0^2(1-2\pi_0)\bar{i}_0^{-1}\ \ ]\mathbf{v}^{(2)*}\ \ (\text{use }\ \mathbf{v}^{(2)} = \mathbf{v}^{(2)*}\bar{i}_0^{-5/2}) \\
&\quad + [\ \ 128\theta_0^5(1-2\pi_0)^2\bar{i}_0^{-1},\ 32\theta_0^3(1-2\pi_0)\bar{i}_0^{-1}, \\
&\qquad 0,\ 8\theta_0\bar{i}_0^{-1},\ 64\theta_0^2(1-2\pi_0)\bar{i}_0^{-1}\ \ ]\mathbf{v}^{(2)*} + O(n^{-1}) \\
&= -\frac{a}{4}\theta_0^{-1}\bar{i}_0^{-1}(1-2\pi_0) + (64+128)\theta_0^5(1-2\pi_0)^2\bar{i}_0^{-1}\frac{3}{256}\theta_0^{-5} \\
&\quad + (16+32)\theta_0^3(1-2\pi_0)\bar{i}_0^{-1}\frac{3}{16}\theta_0^{-4} \\
&\quad + (4+8)\theta_0\bar{i}_0^{-1}\left\{ \frac{1}{4}\theta_0^{-2}(1-2\pi_0) + \frac{1}{2}\theta_0^{-3} \right\} \\
&\quad + (32+64)\theta_0^2(1-2\pi_0)\bar{i}_0^{-1}\left(-\frac{1}{16}\theta_0^{-3} \right) + O(n^{-1}) \\
&= -\frac{a}{4}\theta_0^{-1}\bar{i}_0^{-1}(1-2\pi_0) + \frac{9}{4}(1-2\pi_0)^2\bar{i}_0^{-1} + 9\theta_0^{-1}(1-2\pi_0)\bar{i}_0^{-1} \\
&\quad + 3\theta_0^{-1}(1-2\pi_0)\bar{i}_0^{-1} + 6\theta_0^{-2}\bar{i}_0^{-1} - 6\theta_0^{-1}(1-2\pi_0)\bar{i}_0^{-1} + O(n^{-1}) \\
&= \left\{ \frac{9}{4}(1-2\pi_0)^2 + \left(-\frac{a}{4} + 6 \right)\theta_0^{-1}(1-2\pi_0) + 6\theta_0^{-2} \right\}\bar{i}_0^{-1} + O(n^{-1}),
\end{aligned}
$$

(v) the third part of $2n^2\mathrm{E}_f \underset{(\mathrm{B})}{[} \cdot \underset{(\mathrm{B})}{]} (\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}$ in (S7.9) is

$$2n^2\mathrm{E}_f(n^{-1}2q\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2} = 4n\mathrm{E}_f(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}$$

$$= 4\left\{\frac{1}{2}(1-2\pi_0)+\theta_0^{-1}\right\}\bar{i}_0^{-1/2}(4\theta_0^2\bar{i}_0)^{-1/2} = \{(1-2\pi_0)\theta_0^{-1}+2\theta_0^{-2}\}\bar{i}_0^{-1},$$

(vi) the first half of the fourth term in $\underset{(\mathrm{A})}{[} \cdot \underset{(\mathrm{A})}{]}$ of (S7.9) is equal to the result of (iii) i.e.,

$$n^2\mathrm{E}_f\{2\bar{l}_{\mathrm{ML}}^{(2)}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}\} = -\left\{\frac{3}{2}\theta_0^{-1}(1-2\pi_0)+3\theta_0^{-2}\right\}\bar{i}_0^{-1}+O(n^{-1}),$$

(vii) the second half of the fourth term in $\underset{(\mathrm{A})}{[} \cdot \underset{(\mathrm{A})}{]}$ of (S7.9) is

$$n^2\mathrm{E}_f\{(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)})^2\}$$

$$= \alpha_{\mathrm{ML2}}^{(\mathrm{A})}\mathbf{v}^{(1)}{}'n\,\mathrm{acov}_f(\mathbf{m}_v^{(1)})\mathbf{v}^{(1)} + 2\{n\mathrm{E}_f(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)}\}^2 + O(n^{-1})$$

$$= \alpha_{\mathrm{ML2}}^{(\mathrm{A})}\mathbf{v}^{(1)}{}'\begin{bmatrix} n\,\mathrm{avar}_f(m_v) & n\,\mathrm{cov}_f(m_v,\partial\bar{l}/\partial\theta_0) \\ n\,\mathrm{cov}_f(m_v,\partial\bar{l}/\partial\theta_0) & \gamma \end{bmatrix}\mathbf{v}^{(1)}$$

$$+2\{n\mathrm{E}_f(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)}\}^2 + O(n^{-1})$$

$$= 4\theta_0^2\bar{i}_0\frac{1}{16^2}(\theta_0^2\bar{i}_0)^{-3}(1,\ 8\theta_0)\begin{pmatrix} 16\theta_0^4(1-2\pi_0)^2\bar{i}_0 & 4\theta_0^2(1-2\pi_0)\bar{i}_0 \\ 4\theta_0^2(1-2\pi_0)\bar{i}_0 & \gamma \end{pmatrix}\begin{pmatrix} 1 \\ 8\theta_0 \end{pmatrix}$$

$$+2\left\{\frac{1}{2}(1-2\pi_0)+\theta_0^{-1}\right\}^2\bar{i}_0^{-1}+O(n^{-1})$$

$$= \frac{1}{64}\theta_0^{-4}\bar{i}_0^{-2}\{16\theta_0^4(1-2\pi_0)^2+2\times 32\theta_0^3(1-2\pi_0)+64\theta_0^2\}\bar{i}_0$$

$$+2\left\{\frac{1}{2}(1-2\pi_0)+\theta_0^{-1}\right\}^2\bar{i}_0^{-1}+O(n^{-1})$$

$$
\begin{aligned}
&=\left\{\frac{1}{4}(1-2\pi_0)^2+\theta_0^{-1}(1-2\pi_0)+\theta_0^{-2}\right\}\bar{i}_0^{-1}+2\left\{\frac{1}{2}(1-2\pi_0)+\theta_0^{-1}\right\}^2\bar{i}_0^{-1}\\
&\quad+O(n^{-1})\\
&=\left\{\frac{3}{4}(1-2\pi_0)^2+3\theta_0^{-1}(1-2\pi_0)+3\theta_0^{-2}\right\}\bar{i}_0^{-1}+O(n^{-1}),
\end{aligned}
$$

(viii) the fifth term in $\underset{(\mathrm{A})}{[}\cdot\underset{(\mathrm{A})}{]}$ of (S7.9) is

$$
\begin{aligned}
&-\{2n\mathrm{E}_f(\bar{l}_{\mathrm{ML}}^{(2)})(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\alpha_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})}+(\alpha_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})})^2\}\\
&=-\left[2(-1)(4\theta_0^2\bar{i}_0)^{-1/2}\left\{\frac{1}{2}(1-2\pi_0)+\theta_0^{-1}\right\}\bar{i}_0^{-1/2}+\left\{\frac{1}{2}(1-2\pi_0)+\theta_0^{-1}\right\}^2\bar{i}_0^{-1}\right]\\
&=\left[\theta_0^{-1}\left\{\frac{1}{2}(1-2\pi_0)+\theta_0^{-1}\right\}-\left\{\frac{1}{2}(1-2\pi_0)+\theta_0^{-1}\right\}^2\right]\bar{i}_0^{-1}\\
&=-\left\{\frac{1}{4}(1-2\pi_0)^2+\frac{1}{2}\theta_0^{-1}(1-2\pi_0)\right\}\bar{i}_0^{-1}.
\end{aligned}
$$

Then, from (i) to (viii),

$$
\begin{aligned}
&\kappa_{f2}(t_{\mathrm{W}}^{(\mathrm{A})})=1+n^{-1}\alpha_{(t)\mathrm{W}\Delta 2}^{(\mathrm{A})}+O(n^{-2}),\\
&\alpha_{(t)\mathrm{W}\Delta 2}^{(\mathrm{A})}=\frac{1}{2}\theta_0^{-2}\bar{i}_0^{-1}-\{(1-2\pi_0)^2+2\theta_0^{-1}(1-2\pi_0)\}\bar{i}_0^{-1}+2\\
&\qquad-2\left\{\frac{3}{2}\theta_0^{-1}(1-2\pi_0)+3\theta_0^{-2}\right\}\bar{i}_0^{-1}\\
&\qquad+\left\{\frac{9}{4}(1-2\pi_0)^2+\left(-\frac{a}{4}+6\right)\theta_0^{-1}(1-2\pi_0)+6\theta_0^{-2}\right\}\bar{i}_0^{-1}\\
&\qquad+\{(1-2\pi_0)\theta_0^{-1}+2\theta_0^{-2}\}\bar{i}_0^{-1} \qquad\qquad (\mathrm{S}7.10)\\
&\qquad+\left\{\frac{3}{4}(1-2\pi_0)^2+3\theta_0^{-1}(1-2\pi_0)+3\theta_0^{-2}\right\}\bar{i}_0^{-1}\\
&\qquad-\left\{\frac{1}{4}(1-2\pi_0)^2+\frac{1}{2}\theta_0^{-1}(1-2\pi_0)\right\}\bar{i}_0^{-1}
\end{aligned}
$$

$$
=\left[\left(-1+\frac{9}{4}+\frac{3}{4}-\frac{1}{4}\right)(1-2\pi_0)^2\right.
$$

$$
+\left\{-2\theta_0^{-1}-3\theta_0^{-1}+\left(-\frac{a}{4}+6\right)\theta_0^{-1}+\theta_0^{-1}+3\theta_0^{-1}-\frac{1}{2}\theta_0^{-1}\right\}(1-2\pi_0)
$$

$$
\left.+\frac{1}{2}\theta_0^{-2}-6\theta_0^{-2}+6\theta_0^{-2}+2\theta_0^{-2}+3\theta_0^{-2}\right]\bar{i}_0^{-1}+2
$$

$$
=\left\{\frac{7}{4}(1-2\pi_0)^2+\left(-\frac{a}{4}+\frac{9}{2}\right)\theta_0^{-1}(1-2\pi_0)+\frac{11}{2}\theta_0^{-2}\right\}\bar{i}_0^{-1}+2
$$

($\alpha_{(t)\mathrm{W}\Delta 2}^{(\mathrm{A})}\neq\alpha_{(t)\mathrm{ML}\Delta 2}^{(\mathrm{A})}$ when $a\neq 0$).

$$
\begin{aligned}
\kappa_{f3}(t_{\mathrm{W}}^{(\mathrm{A})}) &= n^{-1/2}\{\alpha_{\mathrm{ML}3}^{(\mathrm{A})}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-3/2}+6n^2\mathrm{E}_f(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)})\}+O(n^{-3/2})\\
&= n^{-1/2}\{\alpha_{\mathrm{ML}3}^{(\mathrm{A})}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-3/2}+6\alpha_{(\Delta t)\mathrm{ML}1}^{(\mathrm{A})}\}+O(n^{-3/2})\\
&= n^{-1/2}\left[-\{8\theta_0^3(1-2\pi_0)+24\theta_0^2\}\bar{i}_0(4\theta_0^2\bar{i}_0)^{-3/2}+6\left\{\frac{1}{2}(1-2\pi_0)+\theta_0^{-1}\right\}\bar{i}_0^{-1/2}\right]\\
&\quad +O(n^{-3/2})\\
&= n^{-1/2}\{(-1+3)(1-2\pi_0)+(-3+6)\theta_0^{-1}\}\bar{i}_0^{-1/2}+O(n^{-3/2})\\
&= n^{-1/2}\{2(1-2\pi_0)+3\theta_0^{-1}\}\bar{i}_0^{-1/2}+O(n^{-3/2}).
\end{aligned}
\tag{S7.11}
$$

$$
\begin{aligned}
\kappa_{f4}(t_{\mathrm{W}}^{(\mathrm{A})}) = n^{-1}\Bigl[_{(\mathrm{A})}\ & \alpha_{\mathrm{ML}4}^{(\mathrm{A})}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-2}+4n^3\mathrm{E}_f\{(\bar{l}_{\mathrm{ML}}^{(1)})^4\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}\}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-3/2}\\
&+12n^3\mathrm{E}_f\{(\bar{l}_{\mathrm{ML}}^{(1)})^3\bar{l}_{\mathrm{ML}}^{(2)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}\}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-3/2}\\
&+6n^3\mathrm{E}_f\{(\bar{l}_{\mathrm{ML}}^{(1)})^4(\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)})^2\}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-1}\\
&+4n^3\mathrm{E}_f\{(\bar{l}_{\mathrm{ML}}^{(1)})^3\bar{l}_{\mathrm{ML}}^{(2)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}+(\bar{l}_{\mathrm{ML}}^{(1)})^4\mathbf{v}^{(2)}{}'\mathbf{m}_v^{(2)}\}(\alpha_{\mathrm{ML}2}^{(\mathrm{A})})^{-3/2}
\end{aligned}
\tag{S7.12}
$$

$$
\begin{aligned}
&+\{4(\alpha^{(\mathrm{A})}_{\mathrm{ML1}}-2q)\alpha^{(\mathrm{A})}_{\mathrm{ML3}}+6\alpha^{(\mathrm{A})}_{\mathrm{ML2}}\alpha^{(\mathrm{A})}_{\mathrm{ML\Delta 2}}+6\alpha^{(\mathrm{A})}_{\mathrm{ML2}}(\alpha^{(\mathrm{A})}_{\mathrm{ML1}}-2q)^2\}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-2}\\
&-4\{\alpha^{(\mathrm{A})}_{(t)\mathrm{ML1}}-2q(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}\}\alpha^{(\mathrm{A})}_{(t)\mathrm{ML3}}\\
&-6\{\alpha^{(\mathrm{A})}_{(t)\mathrm{ML\Delta 2}}-4qn\mathrm{E}_f(\bar{l}^{(1)}_{\mathrm{ML}}\mathbf{m}^{(1)\prime}_{v})\mathbf{v}^{(1)}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}\}\\
&-6\{\alpha^{(\mathrm{A})}_{(t)\mathrm{ML1}}-2q(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}\}^2 \quad \Bigg]_{(\mathrm{A})}+O(n^{-2}),
\end{aligned}
$$

(i) the first term in $\underset{(\mathrm{A})}{[}\ \cdot\ \underset{(\mathrm{A})}{]}$ of (S7.12) is

$$
\begin{aligned}
&\alpha^{(\mathrm{A})}_{\mathrm{ML4}}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-2}\\
&\quad=\{\ 16\theta_0^4(1-6\pi_0+6\pi_0^2)+128\theta_0^3(1-2\pi_0)+192\theta_0^2\ \}\bar{i}_0(4\theta_0^2\bar{i}_0)^{-2}\\
&\quad=\{\ 1-6\pi_0+6\pi_0^2+8\theta_0^{-1}(1-2\pi_0)+12\theta_0^{-2}\}\bar{i}_0^{-1}\\
&\quad=\{(1-2\pi_0)^2-2\bar{i}_0+8\theta_0^{-1}(1-2\pi_0)+12\theta_0^{-2}\}\bar{i}_0^{-1},
\end{aligned}
$$

(ii) the second term in $\underset{(\mathrm{A})}{[}\ \cdot\ \underset{(\mathrm{A})}{]}$ of (S7.12) is

$$
\begin{aligned}
&4n^3\mathrm{E}_f\{(\bar{l}^{(1)}_{\mathrm{ML}})^4\mathbf{v}^{(1)\prime}\mathbf{m}^{(1)}_{v}\}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-3/2}\\
&=4[\ 6\alpha^{(\mathrm{A})}_{\mathrm{ML2}}n^2\mathrm{E}_f\{(\bar{l}^{(1)}_{\mathrm{ML}})^2\mathbf{m}^{(1)\prime}_{v}\}\mathbf{v}^{(1)}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-3/2}\\
&\qquad+4(-2)^3\kappa_{f3}(l_{0j})n\mathrm{E}_f(\bar{l}^{(1)}_{\mathrm{ML}}\mathbf{m}^{(1)\prime}_{v})\mathbf{v}^{(1)}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-3/2}\ ]+O(n^{-1})\\
&=24n^2\mathrm{E}_f\{(\bar{l}^{(1)}_{\mathrm{ML}})^2\mathbf{m}^{(1)\prime}_{v}\}\mathbf{v}^{(1)}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}\\
&\quad-128\theta_0^3(1-2\pi_0)\bar{i}_0\left\{\frac{1}{2}(1-2\pi_0)+\theta_0^{-1}\right\}\bar{i}_0^{-1/2}(4\theta_0^2\bar{i}_0)^{-3/2}+O(n^{-1})\\
&=-12\{(1-2\pi_0)^2+2\theta_0^{-1}(1-2\pi_0)\}\bar{i}_0^{-1}+24\\
&\quad-8(1-2\pi_0)(1-2\pi_0+2\theta_0^{-1})\bar{i}_0^{-1}+O(n^{-1})\\
&=-\{20(1-2\pi_0)^2+40\theta_0^{-1}(1-2\pi_0)\}\bar{i}_0^{-1}+24+O(n^{-1}),
\end{aligned}
$$

(note that the term $24n^2\mathrm{E}_f\{(\bar{l}^{(1)}_{\mathrm{ML}})^2\mathbf{m}^{(1)\prime}_{v}\}\mathbf{v}^{(1)}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}$ is 12 times the result in (ii) for (S7.9) of $\alpha^{(\mathrm{A})}_{(t)\mathrm{ML\Delta 2}}$),

(iii) the third term in $\underset{(\mathrm{A})}{[}\ \cdot\ \underset{(\mathrm{A})}{]}$ of (S7.12) is

$$12n^3\mathrm{E}_f\{(\overline{l}_{\mathrm{ML}}^{(1)})^3\overline{l}_{\mathrm{ML}}^{(2)}\mathbf{v}^{(1)}{}'\mathbf{m}_v^{(1)}\}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}$$

$$=12\underset{(\mathrm{B})}{\Bigg[}\left[3\alpha_{\mathrm{ML2}}^{(\mathrm{A})}n\mathrm{E}_f(\overline{l}_{\mathrm{ML}}^{(2)})+6\lambda^{-1}\left\{n\,\mathrm{cov}_f\left(\overline{l}_{\mathrm{ML}}^{(1)},\frac{\partial\overline{l}}{\partial\theta_0}\right)\right\}^2\right]\mathrm{E}_f(\overline{l}_{\mathrm{ML}}^{(1)}\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)}$$

$$+6\alpha_{\mathrm{ML2}}^{(\mathrm{A})}\lambda^{-1}n\,\mathrm{cov}_f\left(\overline{l}_{\mathrm{ML}}^{(1)},\frac{\partial\overline{l}}{\partial\theta_0}\right)\left\{n\,\mathrm{cov}_f\left(\frac{\partial\overline{l}}{\partial\theta_0},m_v\right),\gamma\right\}\mathbf{v}^{(1)}\underset{(\mathrm{B})}{\Bigg]}(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}$$

$$+O(n^{-1})$$

$$=12\Bigg[\{\,3(4\theta_0^2\overline{i}_0)(-1)+6(-\overline{i}_0^{-1})4(\theta_0\overline{i}_0)^2\,\}\left\{\frac{1}{2}(1-2\pi_0)+\theta_0^{-1}\right\}\overline{i}_0^{-1/2}$$

$$+6(4\theta_0^2\overline{i}_0)(-\overline{i}_0^{-1})(-2\theta_0\overline{i}_0)\{4\theta_0^2(1-2\pi_0)\overline{i}_0,\ \overline{i}_0\}$$

$$\times\left(-\frac{1}{16}\right)(\theta_0^2\overline{i}_0)^{-3/2}(1,\ 8\theta_0)'\Bigg](4\theta_0^2\overline{i}_0)^{-3/2}+O(n^{-1})$$

$$=12\Bigg[\ -\{18(1-2\pi_0)\theta_0^2+36\theta_0\}\overline{i}_0^{1/2}$$

$$+48\theta_0^3\overline{i}_0\left(-\frac{1}{16}\right)(\theta_0^2\overline{i}_0)^{-3/2}\{4\theta_0^2(1-2\pi_0)+8\theta_0\}\overline{i}_0\Bigg](4\theta_0^2\overline{i}_0)^{-3/2}+O(n^{-1})$$

$$=\frac{3}{2}[\ -\{18(1-2\pi_0)\theta_0^2+36\theta_0\}\theta_0^{-3}\overline{i}_0^{-1}$$

$$-3\theta_0^{-3}\overline{i}_0^{-1}\{4\theta_0^2(1-2\pi_0)+8\theta_0\}\ \ ]+O(n^{-1})$$

$$=-\frac{3}{2}\{30\theta_0^{-1}\overline{i}_0^{-1}(1-2\pi_0)+60\theta_0^{-2}\overline{i}_0^{-1}\}+O(n^{-1})$$

$$=-\{45\theta_0^{-1}(1-2\pi_0)+90\theta_0^{-2}\}\overline{i}_0^{-1}+O(n^{-1}),$$

(iv) the fourth term in $\underset{(\mathrm{A})}{[}\ \cdot\ \underset{(\mathrm{A})}{]}$ of (S7.12) is

$$
\begin{aligned}
&6n^3 \mathrm{E}_f \{(\overline{l}^{(1)}_{\mathrm{ML}})^4 (\mathbf{v}^{(1)}{}' \mathbf{m}^{(1)}_v)^2\} (\alpha^{(\mathrm{A})}_{\mathrm{ML}2})^{-1} \\
&= 6[\ 3(\alpha^{(\mathrm{A})}_{\mathrm{ML}2})^2 \mathbf{v}^{(1)}{}' n \,\mathrm{acov}_f (\mathbf{m}^{(1)}_v) \mathbf{v}^{(1)} \\
&\qquad + 12 \alpha^{(\mathrm{A})}_{\mathrm{ML}2} \{n \,\mathrm{cov}_f (\overline{l}^{(1)}_{\mathrm{ML}}, \mathbf{m}^{(1)}_v{}') \mathbf{v}^{(1)}\}^2\ ] (\alpha^{(\mathrm{A})}_{\mathrm{ML}2})^{-1} + O(n^{-1}) \\
&= 18 \alpha^{(\mathrm{A})}_{\mathrm{ML}2} \mathbf{v}^{(1)}{}' n \,\mathrm{acov}_f (\mathbf{m}^{(1)}_v) \mathbf{v}^{(1)} + 72 \{n \,\mathrm{cov}_f (\overline{l}^{(1)}_{\mathrm{ML}}, \mathbf{m}^{(1)}_v{}') \mathbf{v}^{(1)}\}^2 + O(n^{-1}) \\
&= 18 \left\{ \frac{1}{4} (1 - 2\pi_0)^2 + \theta_0^{-1} (1 - 2\pi_0) + \theta_0^{-2} \right\} \overline{i}_0^{\,-1} \\
&\qquad + 36 \times 2 \left\{ \frac{1}{2} (1 - 2\pi_0) + \theta_0^{-1} \right\}^2 \overline{i}_0^{\,-1} + O(n^{-1}) \\
&= \left\{ \left(\frac{9}{2} + 18 \right) (1 - 2\pi_0)^2 + (18 + 72) \theta_0^{-1} (1 - 2\pi_0) + (18 + 72) \theta_0^{-2} \right\} \overline{i}_0^{\,-1} \\
&\qquad + O(n^{-1}) \\
&= \left\{ \frac{45}{2} (1 - 2\pi_0)^2 + 90 \theta_0^{-1} (1 - 2\pi_0) + 90 \theta_0^{-2} \right\} \overline{i}_0^{\,-1} + O(n^{-1})
\end{aligned}
$$

(the two terms with 18 and 72 on the right-hand side of the second equation are 18 and 36 times the corresponding ones in (vii) for (S7.9) of $\alpha^{(\mathrm{A})}_{(t)\mathrm{ML}\Delta 2}$, respectively),

(v) the first half of the fifth term in $\underset{(\mathrm{A})}{[} \cdot \underset{(\mathrm{A})}{]}$ of (S7.12) is one third of the result in (iii) for (S7.12), i.e.,

$$
\begin{aligned}
&4n^3 \mathrm{E}_f \{(\overline{l}^{(1)}_{\mathrm{ML}})^3 \overline{l}^{(2)}_{\mathrm{ML}} \mathbf{v}^{(1)}{}' \mathbf{m}^{(1)}_v\} (\alpha^{(\mathrm{A})}_{\mathrm{ML}2})^{-3/2} \\
&\quad = \frac{1}{3} [-\{45 \theta_0^{-1} (1 - 2\pi_0) + 90 \theta_0^{-2}\} \overline{i}_0^{\,-1}] + O(n^{-1}) \\
&\quad = -\{15 \theta_0^{-1} (1 - 2\pi_0) + 30 \theta_0^{-2}\} \overline{i}_0^{\,-1} + O(n^{-1})
\end{aligned}
$$

(vi) the second half of the fifth term in $\underset{(\mathrm{A})}{[} \cdot \underset{(\mathrm{A})}{]}$ of (S7.12) is

$$4n^3\mathrm{E}_f\{(\overline{l}^{(1)}_{\mathrm{ML}})^4\mathbf{v}^{(2)}{}'\mathbf{m}^{(2)}_v\}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-3/2}$$

$$=4[\ 3(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^2 n\mathrm{E}_f(\mathbf{m}^{(2)}_v{}')\mathbf{v}^{(2)}+6\alpha^{(\mathrm{A})}_{\mathrm{ML2}}n^2\mathrm{acov}_f\{(\overline{l}^{(1)}_{\mathrm{ML}})^2,\,\mathbf{m}^{(2)}_v{}'\}\mathbf{v}^{(2)}\ ]$$

$$\times(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-3/2}+O(n^{-1})$$

$$=12(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{1/2}n\mathrm{E}_f(\mathbf{m}^{(2)}_v{}')\mathbf{v}^{(2)}+24(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-1/2}n^2\mathrm{acov}_f\{(\overline{l}^{(1)}_{\mathrm{ML}})^2,\,\mathbf{m}^{(2)}_v{}'\}\mathbf{v}^{(2)}$$

$$+O(n^{-1})$$

(the first two terms are 6 and 12 times the second and third terms on the right-hand side of the first equation of (iv) for (S7.9) of $\alpha^{(\mathrm{A})}_{(t)\mathrm{ML}\Delta 2}$, respectively)

$$=(6\times 64+12\times 128)(1-2\pi_0)^2\overline{i}_0^{-1}\frac{3}{256}$$

$$+(6\times 16+12\times 32)\theta_0^{-1}(1-2\pi_0)\overline{i}_0^{-1}\frac{3}{16}$$

$$+(6\times 4+12\times 8)\left\{\frac{1}{4}\theta_0^{-1}(1-2\pi_0)+\frac{1}{2}\theta_0^{-2}\right\}\overline{i}_0^{-1}$$

$$-(6\times 32+12\times 64)\theta_0^{-1}(1-2\pi_0)\overline{i}_0^{-1}\frac{1}{16}+O(n^{-1})$$

$$=\frac{45}{2}(1-2\pi_0)^2\overline{i}_0^{-1}+90\theta_0^{-1}(1-2\pi_0)\overline{i}_0^{-1}+30\theta_0^{-1}(1-2\pi_0)\overline{i}_0^{-1}$$

$$+60\theta_0^{-2}\overline{i}_0^{-1}-60\theta_0^{-1}(1-2\pi_0)\overline{i}_0^{-1}+O(n^{-1})$$

$$=\left\{\frac{45}{2}(1-2\pi_0)^2+60\theta_0^{-1}(1-2\pi_0)+60\theta_0^{-2}\right\}\overline{i}_0^{-1}+O(n^{-1}),$$

(vii) the sixth term in $\underset{(\mathrm{A})}{[}\ \cdot\ \underset{(\mathrm{A})}{]}$ of (S7.12) is

$$\{4(\alpha^{(\mathrm{A})}_{\mathrm{ML1}}-2q)\alpha^{(\mathrm{A})}_{\mathrm{ML3}}+6\alpha^{(\mathrm{A})}_{\mathrm{ML2}}\alpha^{(\mathrm{A})}_{\mathrm{ML}\Delta 2}+6\alpha^{(\mathrm{A})}_{\mathrm{ML2}}(\alpha^{(\mathrm{A})}_{\mathrm{ML1}}-2q)^2\}(\alpha^{(\mathrm{A})}_{\mathrm{ML2}})^{-2}$$

$$=[\ 4\times(1-2)\{-8\theta_0^3(1-2\pi_0)-24\theta_0^2\}\overline{i}_0+6(4\theta_0^2\overline{i}_0)2$$

$$+6(4\theta_0^2\overline{i}_0)(1-2)^2\ ](4\theta_0^2\overline{i}_0)^{-2}$$

$$=\left\{2\theta_0^{-1}(1-2\pi_0)+\left(6+3+\frac{3}{2}\right)\theta_0^{-2}\right\}\overline{i}_0^{-1}=\left\{2\theta_0^{-1}(1-2\pi_0)+\frac{21}{2}\theta_0^{-2}\right\}\overline{i}_0^{-1},$$

(viii) the sum of the seventh to ninth terms in $\underset{\text{(A)}}{[}\cdot\underset{\text{(A)}}{]}$ of (S7.12) is

$$
\begin{aligned}
&-4\{\alpha^{\text{(A)}}_{(t)\text{ML}1}-2q(\alpha^{\text{(A)}}_{\text{ML}2})^{-1/2}\}\alpha^{\text{(A)}}_{(t)\text{ML}3}\\
&-6\{\alpha^{\text{(A)}}_{(t)\text{ML}\Delta 2}-4qn\text{E}_f(\overline{l}^{(1)}_{\text{ML}}\mathbf{m}^{(1)}_v{}')\mathbf{v}^{(1)}(\alpha^{\text{(A)}}_{\text{ML}2})^{-1/2}\}-6\{\alpha^{\text{(A)}}_{(t)\text{ML}1}-2q(\alpha^{\text{(A)}}_{\text{ML}2})^{-1/2}\}^2\\
&=-4\left[\left\{\frac{3}{2}\theta_0^{-1}+\frac{1}{2}(1-2\pi_0)\right\}\overline{i}_0^{-1/2}-2(4\theta_0^2\overline{i}_0)^{-1/2}\right]\{2(1-2\pi_0)+3\theta_0^{-1}\}\overline{i}_0^{-1/2}\\
&\quad-6\underset{\text{(B)}}{\Bigg[}\left\{\frac{7}{4}(1-2\pi_0)^2+\left(-\frac{0}{4}+\frac{9}{2}\right)\theta_0^{-1}(1-2\pi_0)+\frac{11}{2}\theta_0^{-2}\right\}\overline{i}_0^{-1}+2\\
&\qquad-4\left\{\frac{1}{2}(1-2\pi_0)+\theta_0^{-1}\right\}\overline{i}_0^{-1/2}(4\theta_0^2\overline{i}_0)^{-1/2}\underset{\text{(B)}}{\Bigg]}\\
&\quad-6\left[\left\{\frac{3}{2}\theta_0^{-1}+\frac{1}{2}(1-2\pi_0)\right\}\overline{i}_0^{-1/2}-2(4\theta_0^2\overline{i}_0)^{-1/2}\right]^2\\
&=-4\left\{\frac{1}{2}\theta_0^{-1}+\frac{1}{2}(1-2\pi_0)\right\}\{2(1-2\pi_0)+3\theta_0^{-1}\}\overline{i}_0^{-1}\\
&\quad-6\left[\left\{\frac{7}{4}(1-2\pi_0)^2+\left(\frac{9}{2}-1\right)\theta_0^{-1}(1-2\pi_0)+\left(\frac{11}{2}-2\right)\theta_0^{-2}\right\}\overline{i}_0^{-1}+2\right]\\
&\quad-6\left\{\frac{1}{2}\theta_0^{-1}+\frac{1}{2}(1-2\pi_0)\right\}^2\overline{i}_0^{-1}\\
&=-\{4(1-2\pi_0)^2+10\theta_0^{-1}(1-2\pi_0)+6\theta_0^{-2}\}\overline{i}_0^{-1}\\
&\quad-\left\{\frac{21}{2}(1-2\pi_0)^2+21\theta_0^{-1}(1-2\pi_0)+21\theta_0^{-2}\right\}\overline{i}_0^{-1}-12\\
&\quad-\left\{\frac{3}{2}(1-2\pi_0)^2+3\theta_0^{-1}(1-2\pi_0)+\frac{3}{2}\theta_0^{-2}\right\}\overline{i}_0^{-1}
\end{aligned}
$$

$$= -\left[\left(4 + \frac{21}{2} + \frac{3}{2}\right)(1-2\pi_0)^2 + (10+21+3)\theta_0^{-1}(1-2\pi_0)\right.$$

$$\left. + \left(6 + 21 + \frac{3}{2}\right)\theta_0^{-2}\right]\bar{i}_0^{-1} - 12$$

$$= -\left\{16(1-2\pi_0)^2 + 34\theta_0^{-1}(1-2\pi_0) + \frac{57}{2}\theta_0^{-2}\right\}\bar{i}_0^{-1} - 12.$$

From (i) to (viii),

$$\kappa_{f4}(t_{\mathrm{W}}^{(\mathrm{A})}) = n^{-1}\Big[_{(\mathrm{A})}\ \{(1-2\pi_0)^2 + 8\theta_0^{-1}(1-2\pi_0) + 12\theta_0^{-2}\}\bar{i}_0^{-1} - 2$$

$$-\{20(1-2\pi_0)^2 + 40\theta_0^{-1}(1-2\pi_0)\}\bar{i}_0^{-1} + 24 - \{45\theta_0^{-1}(1-2\pi_0) + 90\theta_0^{-2}\}\bar{i}_0^{-1}$$

$$+\left\{\frac{45}{2}(1-2\pi_0)^2 + 90\theta_0^{-1}(1-2\pi_0) + 90\theta_0^{-2}\right\}\bar{i}_0^{-1}$$

$$-\{15\theta_0^{-1}(1-2\pi_0) + 30\theta_0^{-2}\}\bar{i}_0^{-1}$$

$$+\left\{\frac{45}{2}(1-2\pi_0)^2 + 60\theta_0^{-1}(1-2\pi_0) + 60\theta_0^{-2}\right\}\bar{i}_0^{-1}$$

$$+\left\{2\theta_0^{-1}(1-2\pi_0) + \frac{21}{2}\theta_0^{-2}\right\}\bar{i}_0^{-1}$$

$$-\left\{16(1-2\pi_0)^2 + 34\theta_0^{-1}(1-2\pi_0) + \frac{57}{2}\theta_0^{-2}\right\}\bar{i}_0^{-1} - 12\ \Big]_{(\mathrm{A})} + O(n^{-2})$$

$$\begin{aligned}
&= n^{-1}\Big[_{\text{(A)}}\Big[_{\text{(B)}}\left(1-20+\frac{45}{2}+\frac{45}{2}-16\right)(1-2\pi_0)^2 \\
&\qquad +(8-40-45+90-15+60+2-34)\theta_0^{-1}(1-2\pi_0) \\
&\qquad +\left(12-90+90-30+60+\frac{21}{2}-\frac{57}{2}\right)\theta_0^{-2}\Big]_{\text{(B)}}\bar{i}_0^{-1} \\
&\qquad -2+24-12\Big]_{\text{(A)}}+O(n^{-2})
\end{aligned} \tag{S7.13}$$

$$\begin{aligned}
&= n^{-1}[\ \{10(1-2\pi_0)^2+26\theta_0^{-1}(1-2\pi_0)+24\theta_0^{-2}\}\bar{i}_0^{-1}+10\ ]+O(n^{-2}) \\
&= n^{-1}\alpha_{(t)\text{ML4}}^{(\text{A})}+O(n^{-2}).
\end{aligned}$$

S7.2.3 A result for estimation of $-2\bar{l}_0^*$

$$\begin{aligned}
& n\,\text{acov}_f\left\{n^{-1}\text{AIC}_{\text{W}},\ \hat{\alpha}_{(t)\text{W1}}^{(\text{A})}+\frac{\hat{\alpha}_{(t)\text{W3}}^{(\text{A})}}{6}(z_{\tilde{\alpha}}^2-1)\right\} \\
&= n\,\text{acov}_f\left\{n^{-1}\text{AIC}_{\text{ML}},\ \hat{\alpha}_{(t)\text{ML1}}^{(\text{A})}+\frac{\hat{\alpha}_{(t)\text{ML3}}^{(\text{A})}}{6}(z_{\tilde{\alpha}}^2-1)\right\} \\
&= n\,\text{acov}_f\left\{\ n^{-1}\text{AIC}_{\text{ML}},\ \hat{\alpha}_{\text{ML1}}^{(\text{A})}(\hat{\alpha}_{\text{ML2}}^{(\text{A})})^{-1/2}+\hat{\alpha}_{(\Delta t)\text{ML1}}^{(\text{A})}z_{\tilde{\alpha}}^2\right. \\
&\qquad\qquad \left.+\frac{1}{6}\hat{\alpha}_{\text{ML3}}^{(\text{A})}(\hat{\alpha}_{\text{ML2}}^{(\text{A})})^{-3/2}(z_{\tilde{\alpha}}^2-1)\right\},
\end{aligned} \tag{S7.14}$$

(i) where the first term on the right-hand side of (S7.14) is

$$\begin{aligned}
& n\,\text{acov}_f\{n^{-1}\text{AIC}_{\text{ML}},\hat{\alpha}_{\text{ML1}}^{(\text{A})}(\hat{\alpha}_{\text{ML2}}^{(\text{A})})^{-1/2}\} \\
&= n\,\text{acov}_f\{\bar{l}_{\text{ML}}^{(1)},(\hat{\alpha}_{\text{ML2}}^{(\text{A})})^{-1/2}\}=n\text{E}_f(\bar{l}_{\text{ML}}^{(1)}\mathbf{m}_v^{(1)}{}')\mathbf{v}^{(1)} \quad (\hat{\alpha}_{\text{ML1}}^{(\text{A})}=\alpha_{\text{ML1}}^{(\text{A})}=1) \\
&= \left\{\frac{1}{2}(1-2\pi_0)+\theta_0^{-1}\right\}\bar{i}_0^{-1/2},
\end{aligned}$$

(ii) the second term on the right-hand side of (S7.14) is

$$n\,\mathrm{acov}_f\{n^{-1}\mathrm{AIC}_{\mathrm{ML}},\, \hat{\alpha}^{(\mathrm{A})}_{(\Delta t)\mathrm{ML1}} z^2_{\tilde{\alpha}}\}$$

$$= n\,\mathrm{acov}_f\{\bar{l}^{(1)}_{\mathrm{ML}}, \widehat{n\mathrm{E}_f(\bar{l}^{(1)}_{\mathrm{ML}}\mathbf{m}^{(1)}_v{}')\mathbf{v}^{(1)}}\} z^2_{\tilde{\alpha}}$$

$$= n\,\mathrm{acov}_f\left[\bar{l}^{(1)}_{\mathrm{ML}}, \left\{\frac{1}{2}(1-2\hat{\pi}_{\mathrm{ML}})+\hat{\theta}^{-1}_{\mathrm{ML}}\right\}\hat{\bar{i}}^{-1/2}_{\mathrm{ML}}\right] z^2_{\tilde{\alpha}}$$

$$= n\,\mathrm{acov}_f\left[-2\theta_0\bar{x}, \left\{\frac{1}{2}(1-2\bar{x})+\left(\log\frac{\bar{x}}{1-\bar{x}}\right)^{-1}\right\}\{\bar{x}(1-\bar{x})\}^{-1/2}\right] z^2_{\tilde{\alpha}}$$

$$= -2\theta_0 n\,\mathrm{var}_f(\bar{x})\left[\left\{-1-\theta_0^{-2}\left(\frac{1}{\pi_0}+\frac{1}{1-\pi_0}\right)\right\}\bar{i}_0^{-1/2}\right.$$

$$\left. + \left\{\frac{1}{2}(1-2\pi_0)+\theta_0^{-1}\right\}\left\{-\frac{1}{2}\bar{i}_0^{-3/2}(1-2\pi_0)\right\}\right] z^2_{\tilde{\alpha}}$$

$$= 2\theta_0\bar{i}_0\left[(1+\theta_0^{-2}\bar{i}_0^{-1})\bar{i}_0^{-1/2}+\left\{\frac{1}{4}(1-2\pi_0)^2\bar{i}_0^{-3/2}+\frac{1}{2}\theta_0^{-1}(1-2\pi_0)\bar{i}_0^{-3/2}\right\}\right] z^2_{\tilde{\alpha}}$$

$$= \left\{\frac{1}{2}(1-2\pi_0)^2\theta_0\bar{i}_0^{-1/2}+(1-2\pi_0)\bar{i}_0^{-1/2}+2(\theta_0\bar{i}_0^{1/2}+\theta_0^{-1}\bar{i}_0^{-1/2})\right\} z^2_{\tilde{\alpha}}$$

(iii) the third term on the right-hand side of (S7.14)

$$n\,\mathrm{acov}_f\left\{n^{-1}\mathrm{AIC}_{\mathrm{ML}},\, \frac{1}{6}\hat{\alpha}^{(\mathrm{A})}_{\mathrm{ML3}}(\hat{\alpha}^{(\mathrm{A})}_{\mathrm{ML2}})^{-3/2}(z^2_{\tilde{\alpha}}-1)\right\}$$

$$= n\,\mathrm{acov}_f[\,\bar{l}^{(1)}_{\mathrm{ML}}, \{-8\hat{\theta}^3_{\mathrm{ML}}(1-2\hat{\pi}_{\mathrm{ML}})-24\hat{\theta}^2_{\mathrm{ML}}\}\hat{\bar{i}}_{\mathrm{ML}}(4\hat{\theta}^2_{\mathrm{ML}}\hat{\bar{i}}_{\mathrm{ML}})^{-3/2}]\frac{z^2_{\tilde{\alpha}}-1}{6}$$

$$= 2\theta_0 n\,\mathrm{acov}_f\left[\bar{x}, \left\{1-2\bar{x}+3\left(\log\frac{\bar{x}}{1-\bar{x}}\right)^{-1}\right\}\{\bar{x}(1-\bar{x})\}^{-1/2}\right]\frac{z^2_{\tilde{\alpha}}-1}{6}$$

$$
\begin{aligned}
&= 2\theta_0 \overline{i}_0 \left[\left\{ -2 - 3\theta_0^{-2} \left(\frac{1}{\pi_0} + \frac{1}{1-\pi_0} \right) \right\} \overline{i}_0^{\,-1/2} \right. \\
&\qquad\qquad \left. + (1 - 2\pi_0 + 3\theta_0^{-1}) \left\{ -\frac{1}{2} \overline{i}_0^{\,-3/2} (1 - 2\pi_0) \right\} \right] \frac{z_{\tilde{\alpha}}^2 - 1}{6} \\
&= -2\theta_0 \overline{i}_0 \left\{ (2 + 3\theta_0^{-2} \overline{i}_0^{\,-1}) \overline{i}_0^{\,-1/2} + \frac{1}{2} (1 - 2\pi_0 + 3\theta_0^{-1})(1 - 2\pi_0) \overline{i}_0^{\,-3/2} \right\} \\
&\qquad \times \frac{z_{\tilde{\alpha}}^2 - 1}{6} \\
&= -\{ \theta_0 \overline{i}_0^{\,-1/2} (1 - 2\pi_0)^2 + 3\overline{i}_0^{\,-1/2} (1 - 2\pi_0) + 4\theta_0 \overline{i}_0^{\,1/2} + 6\theta_0^{-1} \overline{i}_0^{\,-1/2} \} \frac{z_{\tilde{\alpha}}^2 - 1}{6}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
& n \,\mathrm{acov}_f \left\{ n^{-1} \mathrm{AIC}_{\mathrm{W}},\ \hat{\alpha}_{(t)\mathrm{W1}}^{(\mathrm{A})} + \frac{\hat{\alpha}_{(t)\mathrm{W3}}^{(\mathrm{A})}}{6} (z_{\tilde{\alpha}}^2 - 1) \right\} \\
&= \left\{ \frac{1}{2} (1 - 2\pi_0) + \theta_0^{-1} \right\} \overline{i}_0^{\,-1/2} \\
&\ + \left\{ \frac{1}{2} \theta_0 \overline{i}_0^{\,-1/2} (1 - 2\pi_0)^2 + \overline{i}_0^{\,-1/2} (1 - 2\pi_0) + 2\theta_0 \overline{i}_0^{\,1/2} + 2\theta_0^{-1} \overline{i}_0^{\,-1/2}) \right\} z_{\tilde{\alpha}}^2 \\
&- \left\{ \frac{1}{6} \theta_0 \overline{i}_0^{\,-1/2} (1 - 2\pi_0)^2 + \frac{1}{2} \overline{i}_0^{\,-1/2} (1 - 2\pi_0) + \frac{2}{3} \theta_0 \overline{i}_0^{\,1/2} + \theta_0^{-1} \overline{i}_0^{\,-1/2} \right\} \\
&\qquad \times (z_{\tilde{\alpha}}^2 - 1)
\end{aligned}
$$

$$
=\left\{\left(\frac{1}{2}-\frac{1}{6}\right)z_{\tilde{\alpha}}^{2}+\frac{1}{6}\right\}\theta_0\bar{i}_0^{-1/2}(1-2\pi_0)^2
$$

$$
+\left\{\left(1-\frac{1}{2}\right)z_{\tilde{\alpha}}^{2}+\frac{1}{2}+\frac{1}{2}\right\}\bar{i}_0^{-1/2}(1-2\pi_0)+\left\{\left(2-\frac{2}{3}\right)z_{\tilde{\alpha}}^{2}+\frac{2}{3}\right\}\theta_0\bar{i}_0^{1/2}
$$

$$
+\{(2-1)z_{\tilde{\alpha}}^{2}+1+1\}\theta_0^{-1}\bar{i}_0^{-1/2}
$$

$$
=\left(\frac{1}{3}z_{\tilde{\alpha}}^{2}+\frac{1}{6}\right)\theta_0\bar{i}_0^{-1/2}(1-2\pi_0)^2+\left(\frac{1}{2}z_{\tilde{\alpha}}^{2}+1\right)\bar{i}_0^{-1/2}(1-2\pi_0)
$$

$$
+\left(\frac{4}{3}z_{\tilde{\alpha}}^{2}+\frac{2}{3}\right)\theta_0\bar{i}_0^{1/2}+(z_{\tilde{\alpha}}^{2}+2)\theta_0^{-1}\bar{i}_0^{-1/2}. \tag{S7.15}
$$

S7.2.4 Asymptotic cumulants of $n^{-1}\mathrm{AIC}_{\mathrm{ML}}$ after studentization for estimation of $-2\mathrm{E}_f(\hat{\bar{l}}_{\mathrm{W}}^{*})$

$$
t_{\mathrm{W}}^{(\mathrm{A})*}=\frac{n^{1/2}\{n^{-1}\mathrm{AIC}_{\mathrm{W}}+2\mathrm{E}_f(\hat{\bar{l}}_{\mathrm{W}}^{*})\}}{(\hat{v}_{\mathrm{W}}^{(\mathrm{A})})^{1/2}}.
$$

$$
\begin{aligned}
\kappa_{f1}(t_{\mathrm{W}}^{(\mathrm{A})*})&=n^{-1/2}\{\alpha_{(t)\mathrm{W1}}^{(\mathrm{A})}+\lambda^{-1}\gamma(\alpha_{\mathrm{W2}}^{(\mathrm{A})})^{-1/2}\}+O(n^{-3/2})\\
&=n^{-1/2}\{\alpha_{(t)\mathrm{ML1}}^{(\mathrm{A})}+\lambda^{-1}\gamma(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}\}+O(n^{-3/2})\\
&=n^{-1/2}\alpha_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})}+O(n^{-3/2})\\
&=n^{-1/2}\left\{\frac{1}{2}(1-2\pi_0)+\theta_0^{-1}\right\}\bar{i}_0^{-1/2}+O(n^{-3/2})
\end{aligned} \tag{S7.16}
$$

$$
(\alpha_{(t)\mathrm{W1}}^{(\mathrm{A})*}=\alpha_{(t)\mathrm{ML1}}^{(\mathrm{A})*}=\alpha_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})}),
$$

$$
\begin{aligned}
&\kappa_{f2}(t_{\mathrm{W}}^{(\mathrm{A})*})=1+n^{-1}\alpha_{(t)\mathrm{W}\Delta 2}^{(\mathrm{A})*}+O(n^{-2})\\
&=1+n^{-1}\{\alpha_{(t)\mathrm{W}\Delta 2}^{(\mathrm{A})}+2\lambda^{-1}\gamma(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}n\mathrm{E}_f(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{m}_{v}^{(1)}{}')\mathbf{v}^{(1)}\}+O(n^{-2})\\
&=1+n^{-1}\{\alpha_{(t)\mathrm{W}\Delta 2}^{(\mathrm{A})}-2(\alpha_{\mathrm{ML2}}^{(\mathrm{A})})^{-1/2}n\mathrm{E}_f(\bar{l}_{\mathrm{ML}}^{(1)}\mathbf{m}_{v}^{(1)}{}')\mathbf{v}^{(1)}\}+O(n^{-2})
\end{aligned}
$$

$$\begin{aligned}&= 1 + n^{-1}\left[\left\{\frac{7}{4}(1-2\pi_0)^2 + \left(-\frac{a}{4}+\frac{9}{2}\right)\theta_0^{-1}(1-2\pi_0) + \frac{11}{2}\theta_0^{-2}\right\}\bar{i}_0^{-1} + 2\right.\\ &\qquad \left. - 2(4\theta_0^2\bar{i}_0)^{-1/2}\left\{\frac{1}{2}(1-2\pi_0)+\theta_0^{-1}\right\}\bar{i}_0^{-1/2}\right] + O(n^{-2})\\ &= 1 + n^{-1}\left[\left\{\frac{7}{4}(1-2\pi_0)^2 + \left(-\frac{a}{4}+4\right)\theta_0^{-1}(1-2\pi_0) + \frac{9}{2}\theta_0^{-2}\right\}\bar{i}_0^{-1} + 2\right]\\ &\quad + O(n^{-2}).\end{aligned}$$

$$\kappa_{f3}(t_{\mathrm{W}}^{(\mathrm{A})*}) = n^{-1/2}\alpha_{(t)\mathrm{ML3}}^{(\mathrm{A})} + O(n^{-3/2}) \ \ (\alpha_{(t)\mathrm{W3}}^{(\mathrm{A})*} = \alpha_{(t)\mathrm{ML3}}^{(\mathrm{A})*} = \alpha_{(t)\mathrm{ML3}}^{(\mathrm{A})}),$$
$$\kappa_{f4}(t_{\mathrm{W}}^{(\mathrm{A})*}) = n^{-1}\alpha_{(t)\mathrm{ML4}}^{(\mathrm{A})} + O(n^{-2}) \ \ (\alpha_{(t)\mathrm{W4}}^{(\mathrm{A})*} = \alpha_{(t)\mathrm{ML4}}^{(\mathrm{A})*} = \alpha_{(t)\mathrm{ML4}}^{(\mathrm{A})}).$$

S7.2.5 A result for estimation of $-2\mathrm{E}_f(\hat{\bar{l}}_{\mathrm{ML}}^*)$

$$\begin{aligned}& n\,\mathrm{acov}_f\left\{n^{-1}\mathrm{AIC}_{\mathrm{W}},\ \hat{\alpha}_{(t)\mathrm{W1}}^{(\mathrm{A})*} + \frac{\hat{\alpha}_{(t)\mathrm{W3}}^{(\mathrm{A})}}{6}(z_{\tilde{\alpha}}^2-1)\right\}\\ &\quad = n\,\mathrm{acov}_f\left\{n^{-1}\mathrm{AIC}_{\mathrm{ML}},\ \hat{\alpha}_{(\Delta t)\mathrm{ML1}}^{(\mathrm{A})}z_{\tilde{\alpha}}^2 + \frac{1}{6}\hat{\alpha}_{\mathrm{ML3}}^{(\mathrm{A})}(\hat{\alpha}_{\mathrm{ML2}}^{(\mathrm{A})})^{-3/2}(z_{\tilde{\alpha}}^2-1)\right\}\end{aligned}$$

is the sum of the second and third terms of (S7.14) for estimation of $-2\bar{l}_0^*$ i.e.,

$$\begin{aligned}& n\,\mathrm{acov}_f\left\{n^{-1}\mathrm{AIC}_{\mathrm{W}},\ \hat{\alpha}_{(t)\mathrm{W1}}^{(\mathrm{A})*} + \frac{\hat{\alpha}_{(t)\mathrm{W3}}^{(\mathrm{A})}}{6}(z_{\tilde{\alpha}}^2-1)\right\}\\ &= \left(\frac{1}{3}z_{\tilde{\alpha}}^2+\frac{1}{6}\right)\theta_0\bar{i}_0^{-1/2}(1-2\pi_0)^2 + \left(\frac{1}{2}z_{\tilde{\alpha}}^2+\frac{1}{2}\right)\bar{i}_0^{-1/2}(1-2\pi_0)\\ &\quad + \left(\frac{4}{3}z_{\tilde{\alpha}}^2+\frac{2}{3}\right)\theta_0\bar{i}_0^{1/2} + (z_{\tilde{\alpha}}^2+1)\theta_0^{-1}\bar{i}_0^{-1/2}.\end{aligned} \qquad \text{(S7.17)}$$

S7.2.6 Higher-order bias correction of $n^{-1}\mathrm{AIC}_{\mathrm{W}}$

The higher-order bias correction of $n^{-1}\mathrm{AIC}_{\mathrm{W}}$ under correct model specification and canonical parametrization can be done by using $b_2 = c_1$ as

$$n^{-1}\mathrm{AIC}_{\mathrm{W}} - n^{-2}\hat{b}_2 = n^{-1}\mathrm{AIC}_{\mathrm{W}} - n^{-2}\hat{c}_1$$

$$= n^{-1}\mathrm{AIC}_{\mathrm{W}} - n^{-2}(a-1)\{(1-2\hat{\pi}_{\mathrm{ML}})^2\hat{\bar{i}}_{\mathrm{ML}}^{-1} + 2\}$$

$$= n^{-1}\mathrm{AIC}_{\mathrm{W}} + n^{-2}(1-a)[(1-2\bar{x})^2\{\bar{x}(1-\bar{x})\}^{-1} + 2].$$

S7.3 $n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(j)}\,(j = 1,\,2)$

Since $n^{-1}\mathrm{AIC}_{\mathrm{ML}} = n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(j)}\ (j = 1,\,2)$ due to $n^{-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \bar{x}(1-\bar{x})$ in this example, the results in Subsection S7.2 for $n^{-1}\mathrm{AIC}_{\mathrm{ML}}$ also hold for $n^{-1}\mathrm{TIC}_{\mathrm{ML}}^{(j)}\,(j = 1,\,2)$.

References

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