# Supplement I to the paper "Asymptotic cumulants of some information criteria" – Proofs and technical results

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This article is to supplement Ogasawara (2016) with proofs and technical results.

## Appendix. Proofs and technical results
## A. Proofs and associated expressions
## A1. Proof of Theorem 1

We obtain an expression of $b_2$ which is different from that of Konishi and Kitagawa (2003) with $b_1$ being well known. For the expression, we use the formula of the expansion of $\hat{\boldsymbol{\theta}}_{\mathrm{W}} = \boldsymbol{\theta}_{\mathrm{W}}(\mathbf{X}^*)$ given by Ogasawara (2015a, Equation (2.1) (see also 2015b for correction); 2014, Equation (2.4)):

$$\hat{\boldsymbol{\theta}}_{\mathrm{W}} - \boldsymbol{\theta}_0 = -n^{-1}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^* + \sum_{j=1}^{3}\boldsymbol{\Lambda}^{(j)}\mathbf{l}_0^{(j)} - n^{-1}(\hat{\mathbf{L}}_{\mathrm{W}}^{-1}\hat{\mathbf{q}}_{\mathrm{W}}^* - \boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*)_{O_p(n^{-1/2})}$$

$$+ O_p(n^{-2})$$

$$= -n^{-1}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^* + \sum_{j=1}^{3}\boldsymbol{\Lambda}^{(j)}\mathbf{l}_0^{(j)} + n^{-1}\left[\boldsymbol{\Lambda}^{-1}\mathbf{M}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^* - \boldsymbol{\Lambda}^{-1}\frac{\partial\mathbf{q}^*}{\partial\boldsymbol{\theta}_0{}'}\boldsymbol{\Lambda}^{(1)}\mathbf{l}_0^{(1)}\right.$$

$$\left. - \boldsymbol{\Lambda}^{-1}\mathrm{E}_g(\mathbf{J}_0^{(3)})\{(\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*)\otimes\boldsymbol{\Lambda}^{-1}\}\mathbf{l}_0^{(1)}\right] + O_p(n^{-2}) \qquad \text{(A1.1)}$$

$$\equiv -n^{-1}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^* + \sum_{j=1}^{3}\boldsymbol{\Lambda}^{(j)}\mathbf{l}_0^{(j)} + n^{-1}(\mathbf{l}_0^{(\mathrm{W})})_{O_p(n^{-1/2})} + O_p(n^{-2}),$$

where

$$\boldsymbol{\Lambda} = \mathrm{E}_g(\partial^2\overline{l}/\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}) \equiv \mathrm{E}_g(\partial^2\overline{l}/\partial\boldsymbol{\theta}_0\partial\boldsymbol{\theta}_0') = O(1), \quad \mathbf{q}_0^* = \mathbf{q}^*(\boldsymbol{\theta}_0),$$

$$\mathbf{\Lambda}^{(j)} = O(1),\ \mathbf{l}_0^{(j)} = O_p(n^{-j/2})\ (j=1,\ 2,\ 3),$$

$$\hat{\mathbf{L}}_{\mathrm{W}} = \frac{\partial^2 \overline{l}}{\partial \boldsymbol{\theta}\, \partial \boldsymbol{\theta}'}\Big|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}_{\mathrm{W}}} \equiv \frac{\partial^2 \overline{l}}{\partial \hat{\boldsymbol{\theta}}_{\mathrm{W}}\, \partial \hat{\boldsymbol{\theta}}_{\mathrm{W}}'},$$

$$\mathbf{q}_{\mathrm{W}}^* = \mathbf{q}^*(\hat{\boldsymbol{\theta}}_{\mathrm{W}}),\ \mathbf{M} = \frac{\partial^2 \overline{l}}{\partial \boldsymbol{\theta}_0\, \partial \boldsymbol{\theta}_0'} - \mathbf{\Lambda} = O_p(n^{-1/2}),$$

$$\frac{\partial \mathbf{q}^*}{\partial \boldsymbol{\theta}_0'} = \frac{\partial \mathbf{q}^*(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'}\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0},$$

$$\mathbf{J}_0^{(3)} = \frac{\partial^3 \overline{l}}{\partial \boldsymbol{\theta}_0\, (\partial \boldsymbol{\theta}_0')^{<2>}},\ \mathbf{x}^{<k>} = \mathbf{x} \otimes \cdots \otimes \mathbf{x}$$ ($k$ times of $\mathbf{x}$), $\otimes$ denotes the

Kronecker product, and $(\cdot)_{O_p(n^{-1/2})}$ indicates that $(\cdot)$ is of order $O_p(n^{-1/2})$ with other similar expressions.

The term $\sum_{j=1}^{3} \mathbf{\Lambda}^{(j)} \mathbf{l}_0^{(j)}$ in (A1.1) (Ogasawara, 2010, Equation (2.4)) is given from the following expansion:

$$\hat{\boldsymbol{\theta}}_{\mathrm{ML}} - \boldsymbol{\theta}_0 = \sum_{j=1}^{3} \mathbf{\Lambda}^{(j)} \mathbf{l}_0^{(j)} + O_p(n^{-2}), \tag{A1.2}$$

$$\mathbf{\Lambda}^{(1)} \mathbf{l}_0^{(1)} = -\mathbf{\Lambda}^{-1} \frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0},$$

$$\mathbf{\Lambda}^{(2)} \mathbf{l}_0^{(2)} = \mathbf{\Lambda}^{-1} \mathbf{M} \mathbf{\Lambda}^{-1} \frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0} - \frac{1}{2} \mathbf{\Lambda}^{-1} \mathrm{E}_g(\mathbf{J}_0^{(3)}) \left( \mathbf{\Lambda}^{-1} \frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0} \right)^{<2>}$$

$$\mathbf{\Lambda}^{(3)} \mathbf{l}_0^{(3)} = -\mathbf{\Lambda}^{-1} \mathbf{M} \mathbf{\Lambda}^{-1} \mathbf{M} \mathbf{\Lambda}^{-1} \frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0} + \frac{1}{2} \mathbf{\Lambda}^{-1} \mathbf{M} \mathbf{\Lambda}^{-1} \mathrm{E}_g(\mathbf{J}_0^{(3)}) \left( \mathbf{\Lambda}^{-1} \frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0} \right)^{<2>}$$

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$$+\mathbf{\Lambda}^{-1}\mathrm{E}_g(\mathbf{J}_0^{(3)})\left\{\left(\mathbf{\Lambda}^{-1}\mathbf{M}\mathbf{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)\otimes\left(\mathbf{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)\right\}$$

$$-\frac{1}{2}\mathbf{\Lambda}^{-1}\{\mathbf{J}_0^{(3)}-\mathrm{E}_g(\mathbf{J}_0^{(3)})\}\left(\mathbf{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)^{<2>}$$

$$-\frac{1}{2}\mathbf{\Lambda}^{-1}\mathrm{E}_g(\mathbf{J}_0^{(3)})\left[\left(\mathbf{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)\otimes\left\{\mathbf{\Lambda}^{-1}\mathrm{E}_g(\mathbf{J}_0^{(3)})\left(\mathbf{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)^{<2>}\right\}\right]$$

$$+\frac{1}{6}\mathbf{\Lambda}^{-1}\mathrm{E}_g(\mathbf{J}_0^{(4)})\left(\mathbf{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)^{<3>},$$

$$\mathbf{J}_0^{(4)}\equiv\frac{\partial^4\overline{l}}{\partial\boldsymbol{\theta}_0(\partial\boldsymbol{\theta}_0{}')^{<3>}},\ \mathbf{l}_0^{(1)}=\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0},$$

$$\mathbf{l}_0^{(2)}=\left\{\mathrm{v}'(\mathbf{M})\otimes\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0{}'},\left(\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0{}'}\right)^{<2>}\right\}'\equiv(\mathbf{l}_0^{(2-1)}{}',\mathbf{l}_0^{(2-2)}{}')'=O_p(n^{-1}),$$

$$\mathbf{l}_0^{(3)}=\left[\mathrm{v}'(\mathbf{M})^{<2>}\otimes\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0{}'},\mathrm{v}'(\mathbf{M})\otimes\left(\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0{}'}\right)^{<2>}\right.,$$

$$\left.\mathrm{vec}'\{\mathbf{J}_0^{(3)}-\mathrm{E}_g(\mathbf{J}_0^{(3)})\}\otimes\left(\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0{}'}\right)^{<2>},\left(\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0{}'}\right)^{<3>}\right]'$$

$$\equiv(\mathbf{l}_0^{(3-1)}{}',\mathbf{l}_0^{(3-2)}{}',\mathbf{l}_0^{(3-3)}{}',\mathbf{l}_0^{(3-4)}{}')'=O_p(n^{-3/2}),$$

where $\mathbf{\Lambda}^{(2-j)}=O(1)\ (j=1,2)$ and $\mathbf{\Lambda}^{(3-j)}=O(1)\ (j=1,...,4)$ are defined

implicitly by $\mathbf{\Lambda}^{(2)}\mathbf{l}_0^{(2)}=\sum_{j=1}^{2}\mathbf{\Lambda}^{(2-j)}\mathbf{l}_0^{(2-j)}$ and $\mathbf{\Lambda}^{(3)}\mathbf{l}_0^{(3)}=\sum_{j=1}^{4}\mathbf{\Lambda}^{(3-j)}\mathbf{l}_0^{(3-j)}$;

$\mathrm{v}'(\mathbf{M})^{<2>}=[\{\mathrm{v}(\mathbf{M})\}']^{<2>}$; $\mathrm{v}(\cdot)$ is the vectorizing operator taking the

non-duplicated elements of a symmetric matrix in parentheses; and $\mathrm{vec}(\cdot)$ is

the vectorizing operator stacking the columns of a matrix sequentially.

Expand $-2\hat{\bar{l}}_{\mathrm{W}}$ and $-2\hat{\bar{l}}_{\mathrm{W}}^{*}$ as

$$-2\hat{\bar{l}}_{\mathrm{W}} = -2(\bar{l}_0)_{O_p(1)} - 2\sum_{j=1}^{4}\frac{1}{j!}\left\{\frac{\partial^j\bar{l}}{(\partial\boldsymbol{\theta}_0{}')^{<j>}}\right\}_{O_p(1)}\{(\hat{\boldsymbol{\theta}}_{\mathrm{W}}-\boldsymbol{\theta}_0)^{<j>}\}_{O_p(n^{-j/2})}$$
$$+O_p(n^{-5/2}) \tag{A1.3}$$

and

$$-2\hat{\bar{l}}_{\mathrm{W}}^{*} = -2(\bar{l}_0^{*})_{O(1)} - 2\sum_{j=1}^{4}\frac{1}{j!}\left[\mathrm{E}_g\left\{\frac{\partial^j\bar{l}}{(\partial\boldsymbol{\theta}_0{}')^{<j>}}\right\}\right]_{O(1)}\{(\hat{\boldsymbol{\theta}}_{\mathrm{W}}-\boldsymbol{\theta}_0)^{<j>}\}_{O_p(n^{-j/2})}$$
$$+O_p(n^{-5/2}),$$

respectively. Then, recalling $\mathrm{E}_g(\bar{l}_0) = \bar{l}_0^{*}$, we have

$$-2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{W}} - \hat{\bar{l}}_{\mathrm{W}}^{*})$$

$$= -2\mathrm{E}_g\left[\sum_{j=1}^{3}\frac{1}{j!}\left[\frac{\partial^j\bar{l}}{(\partial\boldsymbol{\theta}_0{}')^{<j>}} - \mathrm{E}_g\left\{\frac{\partial^j\bar{l}}{(\partial\boldsymbol{\theta}_0{}')^{<j>}}\right\}\right]_{O_p(n^{-1/2})}(\hat{\boldsymbol{\theta}}_{\mathrm{W}}-\boldsymbol{\theta}_0)^{<j>}\right] \tag{A1.4}$$
$$+O(n^{-3})$$

$$= -2\mathrm{E}_g\left\{\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0{}'}(\hat{\boldsymbol{\theta}}_{\mathrm{W}}-\boldsymbol{\theta}_0)\right\}_{\to O(n^{-2})} - \mathrm{E}_g\{\mathrm{vec}'(\mathbf{M})(\hat{\boldsymbol{\theta}}_{\mathrm{W}}-\boldsymbol{\theta}_0)^{<2>}\}_{\to O(n^{-2})}$$

$$-\frac{1}{3}\mathrm{E}_g\{\mathrm{vec}'\{\mathbf{J}_0^{(3)} - \mathrm{E}_g(\mathbf{J}_0^{(3)})\}(\hat{\boldsymbol{\theta}}_{\mathrm{W}}-\boldsymbol{\theta}_0)^{<3>}\}_{\to O(n^{-2})} + O(n^{-3}),$$

where the terms of $j=4$ in $\sum_{j=1}^{4}(\cdot)$ of (A1.3), when the expectation is taken, are absorbed in the remainder term of order $O(n^{-3})$; and $\mathrm{E}_g(\cdot)_{\to O(n^{-2})}$ indicates that the expectation is taken up to order $O(n^{-2})$.

Let $\boldsymbol{\Gamma} = n\,\mathrm{E}_g\left(\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0}\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0{}'}\right)$. When the model is true, $\boldsymbol{\Gamma} = -\boldsymbol{\Lambda} = \mathbf{I}_0$, where

$\mathbf{I}_0$ is the population Fisher information matrix per observation. Under possible model misspecification, the last three expectations in (A1.4) are given as

$$-2\mathrm{E}_g\left\{\frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0}{}'(\hat{\boldsymbol{\theta}}_{\mathrm{W}}-\boldsymbol{\theta}_0)\right\}$$

$$=-2\mathrm{E}_g\left[\frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0}{}'\{-n^{-1}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*+\sum_{j=1}^{3}\boldsymbol{\Lambda}^{(j)}\mathbf{l}_0^{(j)}+n^{-1}\mathbf{l}_0^{(\mathrm{W})}+O_p(n^{-2})\}\right]$$

$$=\left\{2\mathrm{E}_g\left(\frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0}{}'\boldsymbol{\Lambda}^{-1}\frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0}\right)\right\}_{O(n^{-1})}-\left\{2\mathrm{E}_g\left(\frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0}{}'\boldsymbol{\Lambda}^{(2)}\mathbf{l}_0^{(2)}\right)\right\}_{O(n^{-2})}$$

$$-\left\{2\mathrm{E}_g\left(\frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0}{}'\boldsymbol{\Lambda}^{(3)}\mathbf{l}_0^{(3)}\right)\right\}_{O(n^{-2})}-\left\{2\mathrm{E}_g\left(n^{-1}\frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0}{}'\mathbf{l}_0^{(\mathrm{W})}\right)\right\}_{O(n^{-2})}+O(n^{-3})$$

$$=2n^{-1}\mathrm{tr}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma})-2n^{-2}\left[\underset{(\mathrm{A})}{\sum_{a\geq b}\sum_{c,d=1}^{q}}(\boldsymbol{\Lambda}^{(2-1)})_{(d:ab,\,c)}n^2\mathrm{E}_g\left(m_{ab}\frac{\partial \overline{l}}{\partial \theta_{0c}}\frac{\partial \overline{l}}{\partial \theta_{0d}}\right)\right.\text{(A1.5)}$$

$$+\sum_{a,b,c=1}^{q}(\boldsymbol{\Lambda}^{(2-2)})_{(c:a,b)}n^2\mathrm{E}_g\left(\frac{\partial \overline{l}}{\partial \theta_{0a}}\frac{\partial \overline{l}}{\partial \theta_{0b}}\frac{\partial \overline{l}}{\partial \theta_{0c}}\right)+\sum_{a\geq b}\sum_{c\geq d}\sum_{e,f=1}^{q}(\boldsymbol{\Lambda}^{(3-1)})_{(f:ab,\,cd,\,e)}$$

$$\times\left\{n\,\mathrm{cov}_g(m_{ab},m_{cd})\gamma_{ef}+\sum_{(e,f)}^{2}n\,\mathrm{cov}_g\left(m_{ab},\frac{\partial \overline{l}}{\partial \theta_{0e}}\right)n\,\mathrm{cov}_g\left(m_{cd},\frac{\partial \overline{l}}{\partial \theta_{0f}}\right)\right\}$$

$$+\sum_{a\geq b}\sum_{c,d,e=1}^{q}(\boldsymbol{\Lambda}^{(3-2)})_{(e:ab,\,c,d)}\sum_{(c,d,e)}^{3}n\,\mathrm{cov}_g\left(m_{ab},\frac{\partial \overline{l}}{\partial \theta_{0c}}\right)\gamma_{de}$$

$$+\sum_{a,b,c,d,e,f=1}^{q}(\boldsymbol{\Lambda}^{(3-3)})_{(f:abc,\,d,e)}\sum_{(d,e,f)}^{3}n\,\mathrm{cov}_g\left\{(\mathbf{J}_0^{(3)})_{(a,b,c)},\frac{\partial \overline{l}}{\partial \theta_{0d}}\right\}\gamma_{ef}$$

$$+\sum_{a,b,c,d=1}^{q}(\boldsymbol{\Lambda}^{(3-4)})_{(d:a,b,c)}(\gamma_{ab}\gamma_{cd}+\gamma_{ac}\gamma_{bd}+\gamma_{ad}\gamma_{bc})$$

$$+\sum_{a,b,c=1}^{q}\lambda^{ab}(\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*)_c\,n\,\mathrm{cov}_g\left(\frac{\partial \overline{l}}{\partial \theta_{0a}},m_{bc}\right)+\mathrm{tr}\left(\frac{\partial \mathbf{q}^*}{\partial \boldsymbol{\theta}_0}{}'\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}\boldsymbol{\Lambda}^{-1}\right)$$

$$-\text{tr}[E_g(\mathbf{J}_0^{(3)})\{(\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*)\otimes(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}\boldsymbol{\Lambda}^{-1})\}]\quad\Big]+O(n^{-3})$$
$$\text{(A)}$$

$$\equiv n^{-1}b_1+n^{-2}c_1+O(n^{-3})\ \ (b_1=2\text{tr}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}),\ c_1=-2\ [\ \cdot\ ]\ ),$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(A) (A)}$$

where $(\boldsymbol{\Lambda}^{(2-1)})_{(d:ab,\,c)}$ indicates the element of the $d$-th row and the column corresponding to $(\mathbf{M})_{ab}\equiv m_{ab}$ (the $(a,\,b)$th element of $\mathbf{M}$) and $\partial\bar{l}/\partial(\boldsymbol{\theta}_0)_c\equiv\partial\bar{l}/\partial\theta_{0c}$ of $\boldsymbol{\Lambda}^{(2-1)}$ with $(\cdot)_c$ being the $c$-th element of a vector with other expressions defined similarly;

$$\sum(\cdot)\equiv\sum_{b=1}^{a}\sum_{a=1}^{q}(\cdot),\quad\sum_{e,f=1}^{q}(\cdot)=\sum_{e=1}^{q}\sum_{f=1}^{q}(\cdot);\ \ \text{cov}_g(\cdot)\ \text{is the covariance using the}$$
$$\scriptstyle a\geq b$$

distribution $g(\mathbf{X}^*|\boldsymbol{\zeta}_0)$; $\displaystyle\sum_{(e,f)}^{2}(\cdot)$ is the sum of two symmetric terms with

respect to $e$ and $f$ with $\displaystyle\sum_{(c,d,e)}^{3}(\cdot)$ defined similarly; and $\underset{\text{(A) (A)}}{[\ \cdot\ ]}$ is for ease of finding correspondence;

$$-E_g\{\text{vec}\,'(\mathbf{M})(\hat{\boldsymbol{\theta}}_{\mathrm{W}}-\boldsymbol{\theta}_0)^{<2>}\}$$

$$=-E_g\left[\text{vec}\,'(\mathbf{M})\left\{2(-n^{-1}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*)\otimes\left(-\boldsymbol{\Lambda}^{-1}\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0}\right)+\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0}\right)^{<2>}\right.\right.$$

$$\left.\left.+2\left(-\boldsymbol{\Lambda}^{-1}\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0}\right)\otimes(\boldsymbol{\Lambda}^{(2)}\mathbf{l}_0^{(2)})\right\}\right]$$

$$=-n^{-2}\left[\ 2\sum_{a,b,c=1}^{q}(\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*)_a\lambda^{bc}n\,\text{cov}_g\left(m_{ab},\frac{\partial\bar{l}}{\partial\theta_{0c}}\right)\right.$$
$$\text{(A)}$$

$$\left.+\sum_{a,b,c,d=1}^{q}\lambda^{ac}\lambda^{bd}n^2 E_g\left(m_{ab}\frac{\partial\bar{l}}{\partial\theta_{0c}}\frac{\partial\bar{l}}{\partial\theta_{0d}}\right)\right.$$

$$-2 \sum_{a,b,c=1}^{q} \sum_{d \geq e} \sum_{f=1}^{q} (\mathbf{\Lambda}^{(2-1)})_{(b:de,\,f)}$$

$$\times \lambda^{ac} \left\{ \sum_{(c,f)}^{2} n \operatorname{cov}_g \left( m_{ab}, \frac{\partial \bar{l}}{\partial \theta_{0c}} \right) n \operatorname{cov}_g \left( m_{de}, \frac{\partial \bar{l}}{\partial \theta_{0f}} \right) \right.$$

$$\left. + n \operatorname{cov}_g (m_{ab}, m_{de}) \gamma_{cf} \right\}$$

$$\left. -2 \sum_{a,b,c,d,e=1}^{q} (\mathbf{\Lambda}^{(2-2)})_{(b:d,e)} \lambda^{ac} \sum_{(c,d,e)}^{3} n \operatorname{cov}_g \left( m_{ab}, \frac{\partial \bar{l}}{\partial \theta_{0c}} \right) \gamma_{de} \right]_{(A)} + O(n^{-3}) \qquad \text{(A1.6)}$$

$$\equiv n^{-2} c_2 + O(n^{-3}),$$

$$-\frac{1}{3} \mathrm{E}_g [\operatorname{vec}'\{\mathbf{J}_0^{(3)} - \mathrm{E}_g (\mathbf{J}_0^{(3)})\} (\hat{\mathbf{\theta}}_{\mathrm{W}} - \mathbf{\theta}_0)^{<3>}]$$

$$= -\frac{1}{3} \mathrm{E}_g \left[ \operatorname{vec}'\{\mathbf{J}_0^{(3)} - \mathrm{E}_g (\mathbf{J}_0^{(3)})\} \left( -\mathbf{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \theta_{0d}} \right)^{<3>} \right] + O(n^{-3})$$

$$= n^{-2} \sum_{a,b,c,d,e,f=1}^{q} \lambda^{ad} \lambda^{be} \lambda^{cf} n \operatorname{cov}_g \left\{ (\mathbf{J}_0^{(3)})_{(a,b,c)}, \frac{\partial \bar{l}}{\partial \theta_{0d}} \right\} \gamma_{ef} + O(n^{-3}) \qquad \text{(A1.7)}$$

$$\equiv n^{-2} c_3 + O(n^{-3}),$$

where $\lambda^{bc} = (\mathbf{\Lambda}^{-1})_{bc}$. Then, from (A1.5) to (A1.7) we have (2.9).

## A2. Proof of Corollary 1

Under canonical parametrization in the exponential family, it is known that

$$\frac{\partial^j \bar{l}}{(\partial \mathbf{\theta}_0)^{<j>}} = \mathrm{E}_g \left\{ \frac{\partial^j \bar{l}}{(\partial \mathbf{\theta}_0)^{<j>}} \right\} (j = 2, 3, \dots),$$ which gives $c_1$ of (2.11) from (A1.5) with $\mathbf{M} = \mathbf{O}$ (a zero matrix of an appropriate size) and $\mathbf{J}_0^{(3)} - \mathrm{E}_g (\mathbf{J}_0^{(3)}) = \mathbf{O}$. The results of $c_2 = c_3 = 0$ are derived similarly from (A1.6) and (A1.7) with $\mathbf{M} = \mathbf{O}$ and $\mathbf{J}_0^{(3)} - \mathrm{E}_g (\mathbf{J}_0^{(3)}) = \mathbf{O}$, respectively.

## A3. Proof of Corollary 2

Recalling (A1.2) for $\boldsymbol{\Lambda}^{(2-2)}$ and $\boldsymbol{\Lambda}^{(3-4)}$ in $c_1$ of (2.11), we have

$$
c_1 = -2\left[ \sum_{a,b,c=1}^{q} (\boldsymbol{\Lambda}^{(2-2)})_{(c:a,b)} n^2 \mathrm{E}_g\left( \frac{\partial \bar{l}}{\partial \theta_{0a}} \frac{\partial \bar{l}}{\partial \theta_{0b}} \frac{\partial \bar{l}}{\partial \theta_{0c}} \right) \right.
$$

$$
+ \sum_{a,b,c,d=1}^{q} (\boldsymbol{\Lambda}^{(3-4)})_{(d:a,b,c)} (\gamma_{ab}\gamma_{cd} + \gamma_{ac}\gamma_{bd} + \gamma_{ad}\gamma_{bc})
$$

$$
\left. + \mathrm{tr}\left( \frac{\partial \mathbf{q}^*}{\partial \boldsymbol{\theta}_0{}'} \boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}\boldsymbol{\Lambda}^{-1} \right) - \mathrm{tr}[\mathbf{J}_0^{(3)}\{(\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*) \otimes (\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}\boldsymbol{\Lambda}^{-1})\}] \right]
$$

$$
= -2\left[ \sum_{a,b,c=1}^{q} \left\{ -\frac{1}{2}\boldsymbol{\Lambda}^{-1}\mathbf{J}_0^{(3)}(\boldsymbol{\Lambda}^{-1})^{<2>} \right\}_{(c:a,b)} n^2 \mathrm{E}_g\left( \frac{\partial \bar{l}}{\partial \theta_{0a}} \frac{\partial \bar{l}}{\partial \theta_{0b}} \frac{\partial \bar{l}}{\partial \theta_{0c}} \right) \right.
$$

$$
- \sum_{a,b,c,d=1}^{q} \frac{1}{2}(\boldsymbol{\Lambda}^{-1})_{d\cdot}\mathbf{J}_0^{(3)}\left[ (\boldsymbol{\Lambda}^{-1})_{\cdot a} \otimes [\boldsymbol{\Lambda}^{-1}\mathbf{J}_0^{(3)}\{(\boldsymbol{\Lambda}^{-1})_{\cdot b} \otimes (\boldsymbol{\Lambda}^{-1})_{\cdot c}\}] \right]
$$

$$
\times (\gamma_{ab}\gamma_{cd} + \gamma_{ac}\gamma_{bd} + \gamma_{ad}\gamma_{bc})
$$

$$
+ \sum_{a,b,c,d=1}^{q} \frac{1}{6}\{\boldsymbol{\Lambda}^{-1}\mathbf{J}_0^{(4)}(\boldsymbol{\Lambda}^{-1})^{<3>}\}_{(d:a,b,c)} 3\gamma_{ab}\gamma_{cd}
$$

$$
\left. + \mathrm{tr}\left( \frac{\partial \mathbf{q}^*}{\partial \boldsymbol{\theta}_0{}'} \boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}\boldsymbol{\Lambda}^{-1} \right) - \mathrm{tr}[\mathbf{J}_0^{(3)}\{(\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*) \otimes (\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}\boldsymbol{\Lambda}^{-1})\}] \right]
$$

$$
= -\mathrm{vec}'(\mathbf{J}_0^{(3)})n^2 \mathrm{E}_g\left\{ \left( -\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)^{<3>} \right\}
$$

$$
+ \mathrm{vec}'(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}\boldsymbol{\Lambda}^{-1})\mathbf{J}_0^{(3)}{}'\boldsymbol{\Lambda}^{-1}\mathbf{J}_0^{(3)}\mathrm{vec}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}\boldsymbol{\Lambda}^{-1})
$$

$$
+ 2\mathrm{vec}'(\mathbf{J}_0^{(3)})\{\boldsymbol{\Lambda}^{-1} \otimes (\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}\boldsymbol{\Lambda}^{-1})^{<2>}\}\mathrm{vec}(\mathbf{J}_0^{(3)})
$$

$$
- \mathrm{vec}'(\mathbf{J}_0^{(4)})\mathrm{vec}\{(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}\boldsymbol{\Lambda}^{-1})^{<2>}\} \tag{A3.1}
$$

$$
- 2\mathrm{tr}\left( \frac{\partial \mathbf{q}^*}{\partial \boldsymbol{\theta}_0{}'} \boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}\boldsymbol{\Lambda}^{-1} \right) + 2\mathrm{tr}[\mathbf{J}_0^{(3)}\{(\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*) \otimes (\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}\boldsymbol{\Lambda}^{-1})\}],
$$

where $(\cdot)_{d\cdot}$ is the $d$-th row of a matrix and $(\cdot)_{\cdot a}$ is the $a$-th column of a matrix.

Under correct model specification and canonical parametrization, since

$\partial l_j / \partial \boldsymbol{\theta}_0 = \mathbf{x}^* - \mathrm{E}_f(\mathbf{x}^*)$ and $-\boldsymbol{\Lambda} = \boldsymbol{\Gamma} = \mathbf{I}_0$, (A3.1) becomes

$$c_1 = \boldsymbol{\kappa}_{f3}{}'(\mathbf{x}^*)\boldsymbol{\kappa}_{f3}\left(\mathbf{I}_0^{-1}\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}\right) - \mathrm{vec}'(\mathbf{I}_0^{-1})\mathbf{J}_0^{(3)}{}'\mathbf{I}_0^{-1}\mathbf{J}_0^{(3)}\mathrm{vec}(\mathbf{I}_0^{-1})$$

$$- 2\mathrm{vec}'(\mathbf{J}_0^{(3)})(\mathbf{I}_0^{-1})^{<3>}\mathrm{vec}(\mathbf{J}_0^{(3)}) + \boldsymbol{\kappa}_{f4}{}'(\mathbf{x}^*)\mathrm{vec}\{(\mathbf{I}_0^{-1})^{<2>}\}$$

$$- 2\mathrm{tr}\left(\frac{\partial \mathbf{q}^*}{\partial \boldsymbol{\theta}_0{}'}\mathbf{I}_0^{-1}\right) - 2\mathrm{tr}[\mathbf{J}_0^{(3)}\{(\mathbf{I}_0^{-1}\mathbf{q}_0^*)\otimes \mathbf{I}_0^{-1}\}]$$

$$= \boldsymbol{\kappa}_{f3}{}'(\mathbf{I}_0^{-1/2}\mathbf{x}^*)\boldsymbol{\kappa}_{f3}\left(\mathbf{I}_0^{-1/2}\frac{\partial l_j}{\partial \boldsymbol{\theta}_0}\right)$$

$$- \boldsymbol{\kappa}_{f3}{}'(\mathbf{I}_0^{-1/2}\mathbf{x}^*)[\mathbf{I}_{(q)}\otimes\{\mathrm{vec}(\mathbf{I}_{(q)})\mathrm{vec}'(\mathbf{I}_{(q)})\}]\boldsymbol{\kappa}_{f3}(\mathbf{I}_0^{-1/2}\mathbf{x}^*) \qquad \text{(A3.2)}$$

$$- 2\boldsymbol{\kappa}_{f3}{}'(\mathbf{I}_0^{-1/2}\mathbf{x}^*)\boldsymbol{\kappa}_{f3}(\mathbf{I}_0^{-1/2}\mathbf{x}^*) + \boldsymbol{\kappa}_{f4}{}'(\mathbf{I}_0^{-1/2}\mathbf{x}^*)\mathrm{vec}\{(\mathbf{I}_{(q)})^{<2>}\}$$

$$- 2\mathrm{tr}\left(\frac{\partial \mathbf{q}^*}{\partial \boldsymbol{\theta}_0{}'}\mathbf{I}_0^{-1}\right) + 2\boldsymbol{\kappa}_{f3}{}'(\mathbf{I}_0^{-1/2}\mathbf{x}^*)\{(\mathbf{I}_0^{-1/2}\mathbf{q}_0^*)\otimes\mathrm{vec}(\mathbf{I}_{(q)})\}$$

$$= \boldsymbol{\kappa}_{f3}{}'(\tilde{\mathbf{x}}^*)\boldsymbol{\kappa}_{f3}(\tilde{\mathbf{x}}^*) - \boldsymbol{\kappa}_{f3}{}'(\tilde{\mathbf{x}}^*)[\mathbf{I}_{(q)}\otimes\{\mathrm{vec}(\mathbf{I}_{(q)})\mathrm{vec}'(\mathbf{I}_{(q)})\}]\boldsymbol{\kappa}_{f3}(\tilde{\mathbf{x}}^*)$$

$$- 2\boldsymbol{\kappa}_{f3}{}'(\tilde{\mathbf{x}}^*)\boldsymbol{\kappa}_{f3}(\tilde{\mathbf{x}}^*) + \boldsymbol{\kappa}_{f4}{}'(\tilde{\mathbf{x}}^*)\mathrm{vec}(\mathbf{I}_{(q^2)})$$

$$- 2\mathrm{tr}\left(\frac{\partial \mathbf{q}^*}{\partial \boldsymbol{\theta}_0{}'}\mathbf{I}_0^{-1}\right) + 2\boldsymbol{\kappa}_{f3}{}'(\tilde{\mathbf{x}}^*)\{(\mathbf{I}_0^{-1/2}\mathbf{q}_0^*)\otimes\mathrm{vec}(\mathbf{I}_{(q)})\},$$

which gives (2.12).

## A4. Proof of Corollary 4

Since $\mathbf{J}_0^{(j)} \equiv \dfrac{\partial^j \bar{l}}{\partial \boldsymbol{\theta}_0(\partial \boldsymbol{\theta}_0{}')^{<j-1>}} = \mathrm{E}_g\left\{\dfrac{\partial^j \bar{l}}{\partial \boldsymbol{\theta}_0(\partial \boldsymbol{\theta}_0{}')^{<j-1>}}\right\}(j=2,3,...)$ under

canonical parametrization, the asymptotic expansion using the MLE corresponding to (A1.4) higher than (A1.4) is given only by the first term

$-2\mathrm{E}_g\left\{\dfrac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0{}'}(\hat{\boldsymbol{\theta}}_{\mathrm{ML}}-\boldsymbol{\theta}_0)\right\}$, which is also given only by

$-2\mathrm{E}_g\left\{\dfrac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0{}'}\left(-\boldsymbol{\Lambda}^{-1}\dfrac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right)\right\}$ and $-2\mathrm{E}_g\{h(\mathbf{J}_0^{(3)},\mathbf{J}_0^{(4)},...)\}$, where $h(\cdot)$ is the

sum of multiplicative functions of the powers of the arguments.

In the only non-vanishing term $-2\mathrm{E}_g\left\{\dfrac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}{}'(\hat{\boldsymbol{\theta}}_{\mathrm{ML}}-\boldsymbol{\theta}_0)\right\}$ for the expansion of the left-hand side of (2.15),

$$-2\mathrm{E}_g\left\{\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}{}'\left(-\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right)\right\}=-2\mathrm{tr}\left\{\boldsymbol{\Sigma}\,\mathrm{E}_g\left(\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}{}'\right)\right\}$$

$=-n^{-1}2\mathrm{tr}(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1})=-n^{-1}2q$ under arbitrary distributions as long as $\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}^{-1}$ exist. The remaining terms $-2\mathrm{E}_g\{h(\mathbf{J}_0^{(3)},\mathbf{J}_0^{(4)},...)\}$ vanish when we use the normal distribution even under non-normality since $\mathbf{J}_0^{(j)}=\mathbf{O}\,(j=3,4,...)$ in this case.

An alternative direct proof is given as follows. Let $\mathbf{z}_j\,(j=1,...,n)$ be independent copies of $\mathbf{x}^*$ and $\mathrm{E}_{\mathbf{z}^*}(\cdot)$ denote an expectation over the distribution of $\mathbf{Z}^*$ or $\mathbf{z}_j\,(j=1,...,n)$. Then, by definition,

$$-2\hat{\bar{l}}^*_{\mathrm{ML}}=-2\mathrm{E}_{\mathbf{z}^*}\left[-\frac{n^{-1}}{2}\sum_{j=1}^{n}(\mathbf{z}_j-\bar{\mathbf{x}})'\boldsymbol{\Sigma}^{-1}(\mathbf{z}_j-\bar{\mathbf{x}})-\frac{1}{2}\log\{(2\pi)^q\,|\boldsymbol{\Sigma}|\}\right]$$

$$=\mathrm{tr}(\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma})+(\boldsymbol{\mu}_0-\bar{\mathbf{x}})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_0-\bar{\mathbf{x}})'+\log\{(2\pi)^q\,|\boldsymbol{\Sigma}|\} \qquad \text{(A4.1)}$$

$$=q+(\boldsymbol{\mu}_0-\bar{\mathbf{x}})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_0-\bar{\mathbf{x}})'+\log\{(2\pi)^q\,|\boldsymbol{\Sigma}|\},$$

which gives $-2\mathrm{E}_g(\hat{\bar{l}}^*_{\mathrm{ML}})=(1+n^{-1})q+\log\{(2\pi)^q\,|\boldsymbol{\Sigma}|\}$. On the other hand,

$$-2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{ML}})=-2\mathrm{E}_g\left[-\frac{n^{-1}}{2}\sum_{j=1}^{n}(\mathbf{x}_j-\bar{\mathbf{x}})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}_j-\bar{\mathbf{x}})-\frac{1}{2}\log\{(2\pi)^q\,|\boldsymbol{\Sigma}|\}\right]$$

$$=(1-n^{-1})\mathrm{tr}(\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma})+\log\{(2\pi)^q\,|\boldsymbol{\Sigma}|\} \qquad \text{(A4.2)}$$

$$=(1-n^{-1})q+\log\{(2\pi)^q\,|\boldsymbol{\Sigma}|\}.$$

Consequently, (A4.1) and (A4.2) yield $-2\mathrm{E}_g(\hat{\bar{l}}_{\mathrm{ML}}-\hat{\bar{l}}^*_{\mathrm{ML}})=-n^{-1}2q$.

## A4a. Proof of Corollary 5

Let $c_1^*$ be the sum of the first three terms on the right-hand side of (2.12). Then,

$$c_1 = c_1^* - 2\mathrm{tr}\left(\frac{\partial \mathbf{q}^*}{\partial \boldsymbol{\theta}_0}' \mathbf{I}_0^{-1}\right) + 2\boldsymbol{\kappa}_{f3}'(\tilde{\mathbf{x}}^*)\{(\mathbf{I}_0^{-1/2}\mathbf{q}_0^*) \otimes \mathrm{vec}(\mathbf{I}_{(q)})\} \tag{A4.3}$$

Since

$$\mathbf{q}_0^* = \frac{1}{2}\frac{\partial \log|\mathbf{I}|}{\partial \boldsymbol{\theta}_0} = \frac{1}{2}\frac{\partial \mathrm{vec}'(\mathbf{I})}{\partial \boldsymbol{\theta}_0}\mathrm{vec}(\mathbf{I}_0^{-1})$$
$$= \frac{1}{2}\{\mathbf{I}_0^{1/2} \otimes \mathrm{vec}'(\mathbf{I}_{(q)})\}\boldsymbol{\kappa}_{f3}(\tilde{\mathbf{x}}^*), \tag{A4.4}$$

the last term in (A4.3) becomes

$$2\boldsymbol{\kappa}_{f3}'(\tilde{\mathbf{x}}^*)\{(\mathbf{I}_0^{-1/2}\mathbf{q}_0^*) \otimes \mathrm{vec}(\mathbf{I}_{(q)})\}$$
$$= \boldsymbol{\kappa}_{f3}'(\tilde{\mathbf{x}}^*)[\mathbf{I}_{(q)} \otimes \mathrm{vec}(\mathbf{I}_{(q)})\mathrm{vec}'(\mathbf{I}_{(q)})]\boldsymbol{\kappa}_{f3}(\tilde{\mathbf{x}}^*). \tag{A4.5}$$

On the other hand, the second term on the right-hand side of (A4.3) is

$$-2\mathrm{tr}\left(\frac{\partial \mathbf{q}^*}{\partial \boldsymbol{\theta}_0}' \mathbf{I}_0^{-1}\right)$$

$$= -\mathrm{tr}\left[\underbrace{\left[\left\{\frac{\partial^2}{\partial \boldsymbol{\theta}_0 \partial \boldsymbol{\theta}_0}' \mathrm{vec}'(\mathbf{I})\right\}\mathrm{vec}(\mathbf{I}_0^{-1})\right]}_{(A)} \underbrace{\mathbf{I}_0^{-1}}_{(A)}\right]$$

$$+ \mathrm{tr}\left[\underbrace{\left[\frac{\partial \mathrm{vec}'(\mathbf{I})}{\partial \boldsymbol{\theta}_0}\left\{\mathrm{vec}\left(\mathbf{I}_0^{-1}\frac{\partial \mathbf{I}}{\partial \theta_{01}}\mathbf{I}_0^{-1}\right),...,\mathrm{vec}\left(\mathbf{I}_0^{-1}\frac{\partial \mathbf{I}}{\partial \theta_{0q}}\mathbf{I}_0^{-1}\right)\right\}\right]}_{(B)} \underbrace{\mathbf{I}_0^{-1}}_{(B)}\right] \tag{A4.6}$$

$$= -\boldsymbol{\kappa}_{f4}'(\tilde{\mathbf{x}}^*)\mathrm{vec}(\mathbf{I}_{(q^2)}) + \boldsymbol{\kappa}_{f3}'(\tilde{\mathbf{x}}^*)\boldsymbol{\kappa}_{f3}'(\tilde{\mathbf{x}}^*),$$

where $\left\{\dfrac{\partial^2}{\partial \boldsymbol{\theta}_0 \partial \boldsymbol{\theta}_0}' \mathrm{vec}'(\mathbf{A})\right\}\mathrm{vec}(\mathbf{B}_0) = \displaystyle\sum_{a,b=1}^{q}\dfrac{\partial^2(\mathbf{A})_{ab}}{\partial \boldsymbol{\theta}_0 \partial \boldsymbol{\theta}_0}'(\mathbf{B}_0)_{ab}$ and $(\cdot)_{ab}$ is the

$(a, b)$th element of a matrix. From the definition of $c_1^*$ in (A4.3) (see (2.12)),
we find that the sum of (A4.5) and (A4.6) is $-c_1^*$, which gives
$c_1 = c_1^* - c_1^* = 0$.

## A5. Expressions of $-\boldsymbol{\Lambda}_{\mathbf{M}}^{-1(\Delta)}$, $-\boldsymbol{\Lambda}_{\mathbf{M}}^{-1(\Delta\Delta)}$, $\boldsymbol{\Gamma}_{\mathbf{M}}^{(\Delta)}$ and $\boldsymbol{\Gamma}_{\mathbf{M}}^{(\Delta\Delta)}$

Let $\mathbf{L}_0 = \left( \dfrac{\partial^2 \bar{l}}{\partial \boldsymbol{\theta}_0 \partial \boldsymbol{\theta}_0{}'} \right)_{O_p(1)}$ , then

$$-\hat{\mathbf{L}}_{\mathrm{W}}^{-1} = -\mathbf{L}_0^{-1} + \sum_{j=1}^{q} \mathbf{L}_0^{-1} \frac{\partial \mathbf{L}_0}{\partial (\boldsymbol{\theta}_0)_j} \mathbf{L}_0^{-1} (\hat{\boldsymbol{\theta}}_{\mathrm{W}} - \boldsymbol{\theta}_0)_j$$

$$+ \sum_{j,k=1}^{q} \left\{ -\mathbf{L}_0^{-1} \frac{\partial \mathbf{L}_0}{\partial (\boldsymbol{\theta}_0)_j} \mathbf{L}_0^{-1} \frac{\partial \mathbf{L}_0}{\partial (\boldsymbol{\theta}_0)_k} \mathbf{L}_0^{-1} + \frac{1}{2} \mathbf{L}_0^{-1} \frac{\partial^2 \mathbf{L}_0}{\partial (\boldsymbol{\theta}_0)_j \partial (\boldsymbol{\theta}_0)_k} \mathbf{L}_0^{-1} \right\}$$

$$\times (\hat{\boldsymbol{\theta}}_{\mathrm{W}} - \boldsymbol{\theta}_0)_j (\hat{\boldsymbol{\theta}}_{\mathrm{W}} - \boldsymbol{\theta}_0)_k + O_p(n^{-3/2})$$

$$= -\boldsymbol{\Lambda}^{-1} + \boldsymbol{\Lambda}^{-1} \mathbf{M} \boldsymbol{\Lambda}^{-1} - \boldsymbol{\Lambda}^{-1} \mathbf{M} \boldsymbol{\Lambda}^{-1} \mathbf{M} \boldsymbol{\Lambda}^{-1}$$

$$+ (\boldsymbol{\Lambda}^{-1} - \boldsymbol{\Lambda}^{-1} \mathbf{M} \boldsymbol{\Lambda}^{-1}) \sum_{j=1}^{q} \left[ \mathrm{E}_g \left( \frac{\partial \mathbf{L}_0}{\partial (\boldsymbol{\theta}_0)_j} \right) + \left\{ \frac{\partial \mathbf{L}_0}{\partial (\boldsymbol{\theta}_0)_j} - \mathrm{E}_g \left( \frac{\partial \mathbf{L}_0}{\partial (\boldsymbol{\theta}_0)_j} \right) \right\} \right]$$

$$\times (\boldsymbol{\Lambda}^{-1} - \boldsymbol{\Lambda}^{-1} \mathbf{M} \boldsymbol{\Lambda}^{-1}) \left( -\boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} - n^{-1} \boldsymbol{\Lambda}^{-1} \mathbf{q}_0^* + \boldsymbol{\Lambda}^{(2)} \mathbf{l}_0^{(2)} \right)_j$$

$$+ \sum_{j,k=1}^{q} \left\{ -\boldsymbol{\Lambda}^{-1} \mathrm{E}_g \left( \frac{\partial \mathbf{L}_0}{\partial (\boldsymbol{\theta}_0)_j} \right) \boldsymbol{\Lambda}^{-1} \mathrm{E}_g \left( \frac{\partial \mathbf{L}_0}{\partial (\boldsymbol{\theta}_0)_k} \right) \boldsymbol{\Lambda}^{-1} + \frac{1}{2} \boldsymbol{\Lambda}^{-1} \mathrm{E}_g \left( \frac{\partial^2 \mathbf{L}_0}{\partial (\boldsymbol{\theta}_0)_j \partial (\boldsymbol{\theta}_0)_k} \right) \boldsymbol{\Lambda}^{-1} \right\}$$

$$\times \left( \boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)_j \left( \boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)_k + O_p(n^{-3/2})$$

$$= -\mathbf{\Lambda}^{-1} + \left[ \mathbf{\Lambda}^{-1}\mathbf{M}\mathbf{\Lambda}^{-1} - \mathbf{\Lambda}^{-1}\mathrm{E}_g(\mathbf{J}_0^{(3)}) \left\{ \mathbf{\Lambda}^{-1} \otimes \left( \mathbf{\Lambda}^{-1} \frac{\partial \overline{l}}{\partial \mathbf{\theta}_0} \right) \right\} \right]_{O_p(n^{-1/2})}$$

$$+ \left[ -\mathbf{\Lambda}^{-1}\mathbf{M}\mathbf{\Lambda}^{-1}\mathbf{M}\mathbf{\Lambda}^{-1} + \mathbf{\Lambda}^{-1}\mathbf{M}\mathbf{\Lambda}^{-1}\mathrm{E}_g(\mathbf{J}_0^{(3)}) \left\{ \mathbf{\Lambda}^{-1} \otimes \left( \mathbf{\Lambda}^{-1} \frac{\partial \overline{l}}{\partial \mathbf{\theta}_0} \right) \right\} \right.$$

(A)

$$+ \mathbf{\Lambda}^{-1}\mathrm{E}_g(\mathbf{J}_0^{(3)}) \left\{ (\mathbf{\Lambda}^{-1}\mathbf{M}\mathbf{\Lambda}^{-1}) \otimes \left( \mathbf{\Lambda}^{-1} \frac{\partial \overline{l}}{\partial \mathbf{\theta}_0} \right) \right\}$$

$$- \mathbf{\Lambda}^{-1} \{ \mathbf{J}_0^{(3)} - \mathrm{E}_g(\mathbf{J}_0^{(3)}) \} \left\{ \mathbf{\Lambda}^{-1} \otimes \left( \mathbf{\Lambda}^{-1} \frac{\partial \overline{l}}{\partial \mathbf{\theta}_0} \right) \right\}$$

$$+ \mathbf{\Lambda}^{-1}\mathrm{E}_g(\mathbf{J}_0^{(3)}) \left\{ \mathbf{\Lambda}^{-1} \otimes (-n^{-1}\mathbf{\Lambda}^{-1}\mathbf{q}_0^* + \mathbf{\Lambda}^{(2)}\mathbf{l}_0^{(2)}) \right\}$$

$$- \mathbf{\Lambda}^{-1} \left[ \mathrm{E}_g(\mathbf{J}_0^{(3)}) \left\{ \mathbf{\Lambda}^{-1} \otimes \left( \mathbf{\Lambda}^{-1} \frac{\partial \overline{l}}{\partial \mathbf{\theta}_0} \right) \right\} \right]^2$$

$$\left. + \frac{1}{2}\mathbf{\Lambda}^{-1}\mathrm{E}_g(\mathbf{J}_0^{(4)}) \left\{ \mathbf{\Lambda}^{-1} \otimes \left( \mathbf{\Lambda}^{-1} \frac{\partial \overline{l}}{\partial \mathbf{\theta}_0} \right)^{<2>} \right\} \right]_{(\mathrm{A})O_p(n^{-1})} + O_p(n^{-3/2})$$

(A5.1)

$$\equiv -\mathbf{\Lambda}^{-1} + (-\mathbf{\Lambda}_{\mathbf{M}}^{-1(\Delta)})_{O_p(n^{-1/2})} + (-\mathbf{\Lambda}_{\mathbf{M}}^{-1(\Delta\Delta)})_{O_p(n^{-1})} + O_p(n^{-3/2}).$$

Let

$$\mathbf{G} \equiv \mathbf{G}(\mathbf{\theta}) \equiv \mathbf{G}(\mathbf{\theta}, \mathbf{X}^*) \equiv \left( n^{-1} \sum_{j=1}^{n} \frac{\partial l_j}{\partial \mathbf{\theta}} \frac{\partial l_j}{\partial \mathbf{\theta}'} \right)_{O_p(1)},$$

$$\mathbf{G}_0 \equiv \mathbf{G}(\mathbf{\theta}_0) \equiv \mathbf{G}(\mathbf{\theta}_0, \mathbf{X}^*),$$

$$\mathbf{G}_0 = \mathbf{\Gamma} + (\mathbf{M}_{\mathbf{G}})_{O_p(n^{-1/2})}, \ \mathrm{E}_g(\mathbf{G}_0) = \mathbf{\Gamma}, \ \mathbf{G}_{0(j)}^{(3)} = \partial \mathbf{G}_0 / \partial (\mathbf{\theta}_0)_j,$$

(A5.2)

$$\mathbf{G}_{0(j,k)}^{(4)} = \partial^2 \mathbf{G}_0 / \partial (\mathbf{\theta}_0)_j \partial (\mathbf{\theta}_0)_j \ (j,k = 1,...,q),$$

then

$$\hat{\boldsymbol{\Gamma}}_{\mathrm{W}} = \mathbf{G}(\hat{\boldsymbol{\theta}}_{\mathrm{W}}, \mathbf{X}^*) = (\mathbf{G}_0)_{O_p(1)} + (\hat{\boldsymbol{\Gamma}}_{\mathrm{W}} - \mathbf{G}_0)_{O_p(n^{-1/2})}$$

$$= \boldsymbol{\Gamma} + \mathbf{M_G} + \sum_{j=1}^{q} \mathbf{G}_{0(j)}^{(3)}(\hat{\boldsymbol{\theta}}_{\mathrm{W}} - \boldsymbol{\theta}_0)_j + \frac{1}{2}\sum_{j,k=1}^{q} \mathbf{G}_{0(j,k)}^{(4)}(\hat{\boldsymbol{\theta}}_{\mathrm{W}} - \boldsymbol{\theta}_0)_j(\hat{\boldsymbol{\theta}}_{\mathrm{W}} - \boldsymbol{\theta}_0)_k$$

$$+ O_p(n^{-3/2})$$

$$= \boldsymbol{\Gamma} + \mathbf{M_G} + \sum_{j=1}^{q} \mathbf{G}_{0(j)}^{(3)}\left(-n^{-1}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^* + \sum_{k=1}^{2}\boldsymbol{\Lambda}^{(k)}\mathbf{l}_0^{(k)}\right)_j \qquad \text{(A5.3)}$$

$$+ \frac{1}{2}\sum_{j,k=1}^{q} \mathbf{G}_{0(j,k)}^{(4)}(\boldsymbol{\Lambda}^{(1)}\mathbf{l}_0^{(1)})_j(\boldsymbol{\Lambda}^{(1)}\mathbf{l}_0^{(1)})_k + O_p(n^{-3/2})$$

$$= \boldsymbol{\Gamma} + \left\{\mathbf{M_G} - \sum_{j=1}^{q}\mathrm{E}_g(\mathbf{G}_{0(j)}^{(3)})\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)_j\right\}_{O_p(n^{-1/2})}$$

$$+ \left[\underset{\text{(A)}}{-\sum_{j=1}^{q}\{\mathbf{G}_{0(j)}^{(3)} - \mathrm{E}_g(\mathbf{G}_{0(j)}^{(3)})\}}\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)_j\right.$$

$$+ \sum_{j=1}^{q}\mathrm{E}_g(\mathbf{G}_{0(j)}^{(3)})(-n^{-1}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^* + \boldsymbol{\Lambda}^{(2)}\mathbf{l}_0^{(2)})_j$$

$$\left.+ \frac{1}{2}\sum_{j,k=1}^{q}\mathrm{E}_g(\mathbf{G}_{0(j,k)}^{(4)})\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)_j\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)_k\right]_{\text{(A)}_{O_p(n^{-1})}} + O_p(n^{-3/2})$$

$$\equiv \boldsymbol{\Gamma} + (\boldsymbol{\Gamma}_{\mathbf{M}}^{(\Delta)})_{O_p(n^{-1/2})} + (\boldsymbol{\Gamma}_{\mathbf{M}}^{(\Delta\Delta)})_{O_p(n^{-1})} + O_p(n^{-3/2}).$$

**A6. Actual expressions of** $-\boldsymbol{\Lambda}_{\mathbf{I}}^{-1(\Delta)}, -\boldsymbol{\Lambda}_{\mathbf{I}}^{-1(\Delta\Delta)}, \boldsymbol{\Gamma}_{\mathbf{I}}^{(\Delta)}$ **and** $\boldsymbol{\Gamma}_{\mathbf{I}}^{(\Delta\Delta)}$

Omitting terms with $\mathbf{M}, \mathbf{J}_0^{(3)} - \mathrm{E}_g(\mathbf{J}_0^{(3)}), \mathbf{M_G}$ and $\mathbf{G}_{0(j)}^{(3)} - \mathrm{E}_g(\mathbf{G}_{0(j)}^{(3)})$ in (A5.1) and (A5.3), we obtain

$$\hat{\mathbf{I}}_{\mathrm{W}}^{(-\mathbf{\Lambda})-1} = -\mathbf{\Lambda}^{-1} - \left[ \mathbf{\Lambda}^{-1}\mathrm{E}_g(\mathbf{J}_0^{(3)})\left\{ \mathbf{\Lambda}^{-1} \otimes \left( \mathbf{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right) \right\} \right]_{O_p(n^{-1/2})}$$

$$+ \left[ \quad \mathbf{\Lambda}^{-1}\mathrm{E}_g(\mathbf{J}_0^{(3)})\left\{ \mathbf{\Lambda}^{-1} \otimes (-n^{-1}\mathbf{\Lambda}^{-1}\mathbf{q}_0^* + \mathbf{\Lambda}^{(2)}\mathbf{l}_0^{(2)}) \right\} \right.$$
$$_{(A)}$$

$$- \mathbf{\Lambda}^{-1} \left[ \mathrm{E}_g(\mathbf{J}_0^{(3)})\left\{ \mathbf{\Lambda}^{-1} \otimes \left( \mathbf{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right) \right\} \right]^2$$

$$\left. + \frac{1}{2}\mathbf{\Lambda}^{-1}\mathrm{E}_g(\mathbf{J}_0^{(4)})\left\{ \mathbf{\Lambda}^{-1} \otimes \left( \mathbf{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)^{<2>} \right\} \right] + O_p(n^{-3/2}) \tag{A6.1}$$
$$_{(A)_{O_p(n^{-1})}}$$

$$\equiv -\mathbf{\Lambda}^{-1} + (-\mathbf{\Lambda}_{\mathbf{I}}^{-1(\Delta)})_{O_p(n^{-1/2})} + (-\mathbf{\Lambda}_{\mathbf{I}}^{-1(\Delta\Delta)})_{O_p(n^{-1})} + O_p(n^{-3/2}),$$

$$\hat{\mathbf{I}}_{\mathrm{W}}^{(\boldsymbol{\Gamma})} = \boldsymbol{\Gamma} - \left\{ \sum_{j=1}^{q} \mathrm{E}_g(\mathbf{G}_{0(j)}^{(3)})\left( \mathbf{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)_j \right\}_{O_p(n^{-1/2})}$$

$$+ \left[ \quad \sum_{j=1}^{n} \mathrm{E}_g(\mathbf{G}_{0(j)}^{(3)})(-n^{-1}\mathbf{\Lambda}^{-1}\mathbf{q}_0^* + \mathbf{\Lambda}^{(2)}\mathbf{l}_0^{(2)})_j \right.$$
$$_{(A)}$$

$$\left. + \frac{1}{2}\sum_{j,k=1}^{q} \mathrm{E}_g(\mathbf{G}_{0(j,k)}^{(4)})\left( \mathbf{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)_j \left( \mathbf{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)_k \right] + O_p(n^{-3/2}) \tag{A6.2}$$
$$_{(A)_{O_p(n^{-1})}}$$

$$\equiv \boldsymbol{\Gamma} + (\boldsymbol{\Gamma}_{\mathbf{I}}^{(\Delta)})_{O_p(n^{-1/2})} + (\boldsymbol{\Gamma}_{\mathbf{I}}^{(\Delta\Delta)})_{O_p(n^{-1})} + O_p(n^{-3/2}).$$

## A7. Actual expressions of $d^{(\mathrm{T1})}$ in $\mathrm{E}_g\{2(\mathrm{tr}_{\Delta\Delta}^{(\mathrm{T1})})\}$ and $d^{(\mathrm{T2})}$ in $\mathrm{E}_g\{2(\mathrm{tr}_{\Delta\Delta}^{(\mathrm{T2})})\}$

$$E_g\{2(\text{tr}_{\Delta\Delta}^{(T1)})\}$$

$$= 2E_g\{\text{tr}(-\boldsymbol{\Lambda}_{\mathbf{M}}^{-1(\Delta)}\boldsymbol{\Gamma}_{\mathbf{M}}^{(\Delta)} - \boldsymbol{\Lambda}_{\mathbf{M}}^{-1(\Delta\Delta)}\boldsymbol{\Gamma} - \boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}_{\mathbf{M}}^{(\Delta\Delta)})\}$$

$$= n^{-1}2E_g\,\text{tr}\left[\underbrace{n\left[\boldsymbol{\Lambda}^{-1}\mathbf{M}\boldsymbol{\Lambda}^{-1} - \boldsymbol{\Lambda}^{-1}E_g(\mathbf{J}_0^{(3)})\left\{\boldsymbol{\Lambda}^{-1}\otimes\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)\right\}\right]}_{(A)}\right] \qquad (A7.1)$$

$$\times\left\{\mathbf{M_G} - \sum_{j=1}^{q}E_g(\mathbf{G}_{0(j)}^{(3)})\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)_j\right\}$$

$$+n\underbrace{\left[-\boldsymbol{\Lambda}^{-1}\mathbf{M}\boldsymbol{\Lambda}^{-1}\mathbf{M}\boldsymbol{\Lambda}^{-1} + \boldsymbol{\Lambda}^{-1}\mathbf{M}\boldsymbol{\Lambda}^{-1}E_g(\mathbf{J}_0^{(3)})\left\{\boldsymbol{\Lambda}^{-1}\otimes\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)\right\}\right.}_{(B)}$$

$$+\boldsymbol{\Lambda}^{-1}E_g(\mathbf{J}_0^{(3)})\left\{(\boldsymbol{\Lambda}^{-1}\mathbf{M}\boldsymbol{\Lambda}^{-1})\otimes\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)\right\}$$

$$-\boldsymbol{\Lambda}^{-1}\{\mathbf{J}_0^{(3)} - E_g(\mathbf{J}_0^{(3)})\}\left\{\boldsymbol{\Lambda}^{-1}\otimes\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)\right\}$$

$$+\boldsymbol{\Lambda}^{-1}E_g(\mathbf{J}_0^{(3)})\left\{\boldsymbol{\Lambda}^{-1}\otimes(-n^{-1}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^* + \boldsymbol{\Lambda}^{(2)}\mathbf{l}_0^{(2)})\right\}$$

$$-\boldsymbol{\Lambda}^{-1}\left[E_g(\mathbf{J}_0^{(3)})\left\{\boldsymbol{\Lambda}^{-1}\otimes\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)\right\}\right]^2$$

$$+\frac{1}{2}\boldsymbol{\Lambda}^{-1}E_g(\mathbf{J}_0^{(4)})\left\{\boldsymbol{\Lambda}^{-1}\otimes\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)^{<2>}\right\}\underbrace{\phantom{x}}_{(B)}\Bigg]\boldsymbol{\Gamma}$$

$$-n\boldsymbol{\Lambda}^{-1}\left[\underset{(C)}{-\sum_{j=1}^{q}}\{\mathbf{G}_{0(j)}^{(3)}-\mathrm{E}_g(\mathbf{G}_{0(j)}^{(3)})\}\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)_j\right.$$

$$+\sum_{j=1}^{q}\mathrm{E}_g(\mathbf{G}_{0(j)}^{(3)})(-n^{-1}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*+\boldsymbol{\Lambda}^{(2)}\mathbf{l}_0^{(2)})_j$$

$$\left.+\frac{1}{2}\sum_{j,k=1}^{q}\mathrm{E}_g(\mathbf{G}_{0(j,k)}^{(4)})\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)_j\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)_{k\ (C)}\right]_{(A)}$$

$$=n^{-1}2\left[\underset{(A)}{\phantom{-}}\mathrm{vec}'(\boldsymbol{\Lambda}^{-1})n\mathrm{E}_g(\mathbf{M_G}\otimes\mathbf{M})\mathrm{vec}(\boldsymbol{\Lambda}^{-1})\right.$$

$$-\sum_{a,b,c=1}^{q}(\boldsymbol{\Lambda}^{-1})_a.\mathrm{E}_g(\mathbf{J}_0^{(3)})\{(\boldsymbol{\Lambda}^{-1})_{\bullet b}\otimes(\boldsymbol{\Lambda}^{-1})_{\bullet c}\}n\mathrm{E}_g\left\{\frac{\partial\overline{l}}{\partial\theta_{0c}}(\mathbf{M_G})_{ba}\right\}$$

$$-\sum_{a,b,c,d,e=1}^{q}\lambda^{ac}\lambda^{db}\sum_{j=1}^{q}\mathrm{E}_g(\mathbf{G}_{0(j)}^{(3)})_{ab}\lambda^{je}n\mathrm{E}_g\left(m_{cd}\frac{\partial\overline{l}}{\partial\theta_{0e}}\right)$$

$$\left.+\sum_{a,b,c,d=1}^{q}(\boldsymbol{\Lambda}^{-1})_a.\mathrm{E}_g(\mathbf{J}_0^{(3)})\{(\boldsymbol{\Lambda}^{-1})_{\bullet b}\otimes(\boldsymbol{\Lambda}^{-1})_{\bullet c}\}\mathrm{E}_g(\mathbf{G}_{0(j)}^{(3)})_{ab}\lambda^{jd}\gamma_{cd}\right]$$

$$+\left[\underset{(B)}{\phantom{-}}-\mathrm{vec}'(\boldsymbol{\Lambda}^{-1})n\mathrm{E}_g(\mathbf{M}^{<2>})\mathrm{vec}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}\boldsymbol{\Lambda}^{-1})\right.$$

$$+2\sum_{a,b,c=1}^{q}(\boldsymbol{\Lambda}^{-1})_a.\mathrm{E}_g(\mathbf{J}_0^{(3)})\{(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}\boldsymbol{\Lambda}^{-1})_{\bullet b}\otimes(\boldsymbol{\Lambda}^{-1})_{\bullet c}\}n\mathrm{E}_g\left(m_{ab}\frac{\partial\overline{l}}{\partial\theta_{0c}}\right)$$

$$-\sum_{a=1}^{q}\mathrm{tr}\left[n\mathrm{E}_g\left\{\{\mathbf{J}_0^{(3)}-\mathrm{E}_g(\mathbf{J}_0^{(3)})\}\frac{\partial\overline{l}}{\partial\theta_{0a}}\right\}\{(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}\boldsymbol{\Lambda}^{-1})\otimes(\boldsymbol{\Lambda}^{-1})_{\bullet a}\}\right]$$

$$+\mathrm{tr}[\mathrm{E}_g(\mathbf{J}_0^{(3)})\{(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}\boldsymbol{\Lambda}^{-1})\otimes\boldsymbol{\alpha}_{\mathrm{W1}}\}]$$

$$-\sum_{a,b=1}^{q} \text{tr}[\boldsymbol{\Gamma}\boldsymbol{\Lambda}^{-1}\text{E}_g(\mathbf{J}_0^{(3)})\{\boldsymbol{\Lambda}^{-1}\otimes(\boldsymbol{\Lambda}^{-1})_{\bullet a}\}\text{E}_g(\mathbf{J}_0^{(3)})\{\boldsymbol{\Lambda}^{-1}\otimes(\boldsymbol{\Lambda}^{-1})_{\bullet b}\}]\gamma_{ab}$$

$$+\frac{1}{2}\text{vec}'\text{E}_g(\mathbf{J}_0^{(4)})\{\text{vec}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}\boldsymbol{\Lambda}^{-1})\}^{<2>} \Bigg]$$
(B)

$$-\Bigg[ -\sum_{a,b=1}^{q} \text{tr}\, n\text{E}_g\Bigg[\boldsymbol{\Lambda}^{-1}\{\mathbf{G}_{0(a)}^{(3)}-\text{E}_g(\mathbf{G}_{0(a)}^{(3)})\}\frac{\partial\bar{l}}{\partial\theta_{0b}}\Bigg]\lambda^{ab}$$
(C)

$$+\sum_{j=1}^{q}\text{tr}\{\boldsymbol{\Lambda}^{-1}\text{E}_g(\mathbf{G}_{0(j)}^{(3)})(\boldsymbol{\alpha}_{\text{W1}})_j\}+\frac{1}{2}\sum_{j,k=1}^{q}\text{tr}\{\boldsymbol{\Lambda}^{-1}\text{E}_g(\mathbf{G}_{0(j,k)}^{(4)})(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}\boldsymbol{\Lambda}^{-1})_{jk}\}\Bigg]\Bigg]$$
(C)   (A)

$$\equiv n^{-1}d^{(\text{T1})},$$

where $n^{-1}\boldsymbol{\alpha}_{\text{W1}}\equiv-n^{-1}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^{*}+\text{E}_g(\boldsymbol{\Lambda}^{(2)}\mathbf{l}_0^{(2)})$ is the vector of the asymptotic

biases of $\hat{\boldsymbol{\theta}}_{\text{W}}$ up to order $O(n^{-1})$ under possible model misspecification.

On the other hand,

$$\text{E}_g\{2(\text{tr}_{\Delta\Delta}^{(\text{T2})})\}$$

$$=2\text{E}_g\{\text{tr}(-\boldsymbol{\Lambda}_{\mathbf{I}}^{-1(\Delta)}\boldsymbol{\Gamma}_{\mathbf{I}}^{(\Delta)}-\boldsymbol{\Lambda}_{\mathbf{I}}^{-1(\Delta\Delta)}\boldsymbol{\Gamma}-\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}_{\mathbf{I}}^{(\Delta\Delta)})\}$$

$$=n^{-1}2\text{E}_g\text{tr}\Bigg[\, n\Bigg[-\boldsymbol{\Lambda}^{-1}\text{E}_g(\mathbf{J}_0^{(3)})\bigg\{\boldsymbol{\Lambda}^{-1}\otimes\bigg(\boldsymbol{\Lambda}^{-1}\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0}\bigg)\bigg\}$$
(A)

$$\times\bigg\{-\sum_{j=1}^{q}\text{E}_g(\mathbf{G}_{0(j)}^{(3)})\bigg(\boldsymbol{\Lambda}^{-1}\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0}\bigg)_j\bigg\}\Bigg]$$

$$+n\Bigg[\,\boldsymbol{\Lambda}^{-1}\text{E}_g(\mathbf{J}_0^{(3)})\big\{\boldsymbol{\Lambda}^{-1}\otimes(-n^{-1}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^{*}+\boldsymbol{\Lambda}^{(2)}\mathbf{l}_0^{(2)})\big\}$$   (A7.2)
(B)

$$-\boldsymbol{\Lambda}^{-1}\Bigg[\text{E}_g(\mathbf{J}_0^{(3)})\bigg\{\boldsymbol{\Lambda}^{-1}\otimes\bigg(\boldsymbol{\Lambda}^{-1}\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0}\bigg)\bigg\}\Bigg]^2\Bigg]$$

$$+\frac{1}{2}\mathbf{\Lambda}^{-1}\mathrm{E}_g(\mathbf{J}_0^{(4)})\left\{\mathbf{\Lambda}^{-1}\otimes\left(\mathbf{\Lambda}^{-1}\frac{\partial\bar{l}}{\partial\boldsymbol{\theta}_0}\right)^{<2>}\right\}\Bigg]_{(B)}\mathbf{\Gamma}\Bigg]_{(A)}$$

$$-n^{-1}2\mathrm{E}_g\{\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma}_{\mathbf{I}}^{(\Delta\Delta)})\}$$

$$=n^{-1}2\Bigg[\underset{(A)}{\sum_{a,b,c,d=1}^{q}(\mathbf{\Lambda}^{-1})_{a\cdot}\mathrm{E}_g(\mathbf{J}_0^{(3)})\{(\mathbf{\Lambda}^{-1})_{\cdot b}\otimes(\mathbf{\Lambda}^{-1})_{\cdot c}\}\sum_{j=1}^{q}\mathrm{E}_g(\mathbf{G}_{0(j)}^{(3)})_{ab}\lambda^{jd}\gamma_{cd}}$$

$$+\underset{(B)}{\Bigg[\;\mathrm{tr}[\mathrm{E}_g(\mathbf{J}_0^{(3)})\{(\mathbf{\Lambda}^{-1}\mathbf{\Gamma}\mathbf{\Lambda}^{-1})\otimes\boldsymbol{\alpha}_{\mathrm{W1}}\}]}$$

$$-\sum_{a,b=1}^{q}\mathrm{tr}[\mathbf{\Gamma}\mathbf{\Lambda}^{-1}\mathrm{E}_g(\mathbf{J}_0^{(3)})\{\mathbf{\Lambda}^{-1}\otimes(\mathbf{\Lambda}^{-1})_{\cdot a}\}\mathrm{E}_g(\mathbf{J}_0^{(3)})\{\mathbf{\Lambda}^{-1}\otimes(\mathbf{\Lambda}^{-1})_{\cdot b}\}\gamma_{ab}$$

$$+\frac{1}{2}\mathrm{vec}'\mathrm{E}_g(\mathbf{J}_0^{(4)})\{\mathrm{vec}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma}\mathbf{\Lambda}^{-1})\}^{<2>}\Bigg]_{(B)}$$

$$-\underset{(C)}{\Bigg[\sum_{j=1}^{q}\mathrm{tr}\{\mathbf{\Lambda}^{-1}\mathrm{E}_g(\mathbf{G}_{0(j)}^{(3)})(\boldsymbol{\alpha}_{\mathrm{W1}})_j\}}+\frac{1}{2}\sum_{j,k=1}^{q}\mathrm{tr}\{\mathbf{\Lambda}^{-1}\mathrm{E}_g(\mathbf{G}_{0(j,k)}^{(4)})\}(\mathbf{\Lambda}^{-1}\mathbf{\Gamma}\mathbf{\Lambda}^{-1})_{jk}\underset{(C)}{\Bigg]}\;\underset{(A)}{\Bigg]}$$

$$\equiv n^{-1}d^{(\mathrm{T2})}.$$

## A8. The derivation and actual expressions of $(\bar{l}_{\mathrm{ML}}^{(j)})_{O_p(n^{-j/2})}\;(j=1,...,4)$

The five terms up to order $O_p(n^{-2})$ in the last expression of (4.3) are further expanded one by one as follows:

(i)
$$-2(\bar{l}_0)_{O_p(1)}=-2\mathrm{E}_g(\bar{l}_0)-2\{\bar{l}_0-\mathrm{E}_g(\bar{l}_0)\}$$
$$\equiv-2(\bar{l}_0^*)_{O(1)}-2(\bar{l}_0-\bar{l}_0^*)_{O_p(n^{-1/2})},$$

(ii)

$$-2\frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0{}'}\left(-n^{-1}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^* + \sum_{k=1}^{3}\boldsymbol{\Lambda}^{(k)}\mathbf{l}_0^{(k)} + n^{-1}\mathbf{l}_0^{(W)}\right)$$

$$= 2\left(n^{-1}\frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0{}'}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*\right)_{O_p(n^{-3/2})} + 2\left(\frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0{}'}\boldsymbol{\Lambda}^{-1}\frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0}\right)_{O_p(n^{-1})}$$

$$-2\left(\frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0{}'}\boldsymbol{\Lambda}^{(2)}\mathbf{l}_0^{(2)}\right)_{O_p(n^{-3/2})} - 2\left(\frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0{}'}\boldsymbol{\Lambda}^{(3)}\mathbf{l}_0^{(3)}\right)_{O_p(n^{-2})} \tag{A8.1}$$

$$-2\left(n^{-1}\frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0{}'}\mathbf{l}_0^{(W)}\right)_{O_p(n^{-2})},$$

(iii)

$$-\left\{\frac{\partial^2 \overline{l}}{(\partial \boldsymbol{\theta}_0{}')^{<2>}}\right\}_{O_p(1)}\left(-n^{-1}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^* + \sum_{k=1}^{3}\boldsymbol{\Lambda}^{(k)}\mathbf{l}_0^{(k)} + n^{-1}\mathbf{l}_0^{(W)}\right)^{<2>}$$

$$= -\text{vec}'\{\boldsymbol{\Lambda} + (\mathbf{M})_{O_p(n^{-1/2})}\}\left(-n^{-1}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^* + \sum_{k=1}^{3}\boldsymbol{\Lambda}^{(k)}\mathbf{l}_0^{(k)} + n^{-1}(\mathbf{l}_0^{(W)})\right)^{<2>}$$

$$= -\{n^{-2}\text{vec}'(\boldsymbol{\Lambda})(\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*)^{<2>}\}_{O(n^{-2})}$$

$$-2\left[n^{-1}\text{vec}'(\boldsymbol{\Lambda})\left\{(\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*)\otimes\left(\boldsymbol{\Lambda}^{-1}\frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0}\right)\right\}\right]_{O_p(n^{-3/2})}$$

$$+2[n^{-1}\text{vec}'(\boldsymbol{\Lambda})\{(\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*)\otimes(\boldsymbol{\Lambda}^{(2)}\mathbf{l}_0^{(2)})\}]_{O_p(n^{-2})} - \left\{\text{vec}'(\boldsymbol{\Lambda})\left(\boldsymbol{\Lambda}^{-1}\frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0}\right)^{<2>}\right\}_{O_p(n^{-1})}$$

$$+2\left[\text{vec}'(\boldsymbol{\Lambda})\left\{\left(\boldsymbol{\Lambda}^{-1}\frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0}\right)\otimes(\boldsymbol{\Lambda}^{(2)}\mathbf{l}_0^{(2)})\right\}\right]_{O_p(n^{-3/2})}$$

$$+2\left[\text{vec}'(\boldsymbol{\Lambda})\left\{\left(\boldsymbol{\Lambda}^{-1}\frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0}\right)\otimes(\boldsymbol{\Lambda}^{(3)}\mathbf{l}_0^{(3)})\right\}\right]_{O_p(n^{-2})}$$

$$-\{\text{vec}'(\boldsymbol{\Lambda})(\boldsymbol{\Lambda}^{(2)}\mathbf{l}_0^{(2)})^{<2>}\}_{O_p(n^{-2})} + 2\left[n^{-1}\text{vec}'(\boldsymbol{\Lambda})\left\{\left(\boldsymbol{\Lambda}^{-1}\frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0}\right)\otimes\mathbf{l}_0^{(W)}\right\}\right]_{O_p(n^{-2})}$$

$$
-2\left[n^{-1}\text{vec}'(\mathbf{M})\left\{(\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*)\otimes\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)\right\}\right]_{O_p(n^{-2})}
$$

$$
-\left\{\text{vec}'(\mathbf{M})\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)^{<2>}\right\}_{O_p(n^{-3/2})}+2\left[\text{vec}'(\mathbf{M})\left\{\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)\otimes(\boldsymbol{\Lambda}^{(2)}\mathbf{l}_0^{(2)})\right\}\right]_{O_p(n^{-2})},
$$

(iv)

$$
-\frac{1}{3}\frac{\partial^3\overline{l}}{(\partial\boldsymbol{\theta}_0')^{<3>}}\left(-n^{-1}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*+\sum_{k=1}^{2}\boldsymbol{\Lambda}^{(k)}\mathbf{l}_0^{(k)}\right)^{<3>}
$$

$$
=-\frac{1}{3}\text{vec}'[\mathrm{E}_g(\mathbf{J}_0^{(3)})+\{\mathbf{J}_0^{(3)}-\mathrm{E}_g(\mathbf{J}_0^{(3)})\}_{O_p(n^{-1/2})}]\left(-n^{-1}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*+\sum_{k=1}^{2}\boldsymbol{\Lambda}^{(k)}\mathbf{l}_0^{(k)}\right)^{<3>}
$$

$$
=\left[n^{-1}\text{vec}'\{\mathrm{E}_g(\mathbf{J}_0^{(3)})\}\left\{(\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*)\otimes\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)^{<2>}\right\}\right]_{O_p(n^{-2})}
$$

$$
+\frac{1}{3}\left[\text{vec}'\{\mathrm{E}_g(\mathbf{J}_0^{(3)})\}\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)^{<3>}\right]_{O_p(n^{-3/2})}
$$

$$
-\left[\text{vec}'\{\mathrm{E}_g(\mathbf{J}_0^{(3)})\}\left\{\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)^{<2>}\otimes(\boldsymbol{\Lambda}^{(2)}\mathbf{l}_0^{(2)})\right\}\right]_{O_p(n^{-2})}
$$

$$
+\frac{1}{3}\left[\text{vec}'\{\mathbf{J}_0^{(3)}-\mathrm{E}_g(\mathbf{J}_0^{(3)})\}\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)^{<3>}\right]_{O_p(n^{-2})},
$$

(v)

$$
-\frac{1}{12}\mathrm{E}_g\left\{\frac{\partial^4\overline{l}}{(\partial\boldsymbol{\theta}_0')^{<4>}}\right\}(\boldsymbol{\Lambda}^{(1)}\mathbf{l}_0^{(1)})^{<4>}=-\frac{1}{12}\left[\text{vec}'\{\mathrm{E}_g(\mathbf{J}_0^{(4)})\}\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)^{<4>}\right]_{O_p(n^{-2})}.
$$

Using $\dfrac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0'}\boldsymbol{\Lambda}^{-1}\dfrac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}=\text{vec}'(\boldsymbol{\Lambda})\left(\boldsymbol{\Lambda}^{-1}\dfrac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)^{<2>}$ and similar results in (A8.1),

(4.3) becomes

$$-2\hat{\bar{l}}_{\text{W}} = -2(\bar{l}_0^*)_{O(1)} - 2(\bar{l}_0 - \bar{l}_0^*)_{O_p(n^{-1/2})} + \left(\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'} \boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right)_{O_p(n^{-1})}$$

$$+\Bigg[\underset{(A)}{\phantom{x}} 2n^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^* - 2\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'}\boldsymbol{\Lambda}^{(2)}\mathbf{l}_0^{(2)}$$

$$-2n^{-1}\text{vec}'(\boldsymbol{\Lambda})\left\{(\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*)\otimes\left(\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right)\right\}$$

$$+2\text{vec}'(\boldsymbol{\Lambda})\left\{\left(\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right)\otimes(\boldsymbol{\Lambda}^{(2)}\mathbf{l}_0^{(2)})\right\} - \text{vec}'(\mathbf{M})\left(\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right)^{<2>} \quad (A8.2)$$

$$+\frac{1}{3}\text{vec}'\{\text{E}_g(\mathbf{J}_0^{(3)})\}\left(\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right)^{<3>} \Bigg]_{(A)O_p(n^{-3/2})}$$

$$-\{n^{-2}\text{vec}'(\boldsymbol{\Lambda})(\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*)^{<2>}\}_{O(n^{-2})}$$

$$+\Bigg[\underset{(B)\ (1)\dots\dots\dots\dots}{\phantom{x}} -2\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'}\boldsymbol{\Lambda}^{(3)}\mathbf{l}_0^{(3)} - \underset{(2)\dots\dots\dots\dots}{2n^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'}\mathbf{l}_0^{(\text{W})}} + \underset{(3)\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots}{2n^{-1}\text{vec}'(\boldsymbol{\Lambda})\{(\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*)\otimes(\boldsymbol{\Lambda}^{(2)}\mathbf{l}_0^{(2)})\}}$$

$$+ \underset{-(1)\dots\dots\dots\dots\dots\dots\dots}{2\text{vec}'(\boldsymbol{\Lambda})\left\{\left(\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right)\otimes(\boldsymbol{\Lambda}^{(3)}\mathbf{l}_0^{(3)})\right\}} \ \ -\underset{(4)\dots\dots\dots\dots\dots\dots}{\text{vec}'(\boldsymbol{\Lambda})(\boldsymbol{\Lambda}^{(2)}\mathbf{l}_0^{(2)})^{<2>}}$$

$$+ \underset{-(2)\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots}{2n^{-1}\text{vec}'(\boldsymbol{\Lambda})\left\{\left(\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right)\otimes\mathbf{l}_0^{(\text{W})}\right\}} \ \ -\underset{-(3)\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots}{2n^{-1}\text{vec}'(\mathbf{M})\left\{(\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*)\otimes\left(\boldsymbol{\Lambda}^{-1}\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0}\right)\right\}}$$

$$+2\,\text{vec}'(\mathbf{M})\left\{\left(\mathbf{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)\otimes(\mathbf{\Lambda}^{(2)}\mathbf{l}_0^{(2)})\right\}$$

$-2\times(4)\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

$$+n^{-1}\text{vec}'\{\mathrm{E}_g(\mathbf{J}_0^{(3)})\}\left\{(\mathbf{\Lambda}^{-1}\mathbf{q}_0^*)\otimes\left(\mathbf{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)^{<2>}\right\}$$

$-(3)\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

$$-\text{vec}'\{\mathrm{E}_g(\mathbf{J}_0^{(3)})\}\left\{\left(\mathbf{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)^{<2>}\otimes(\mathbf{\Lambda}^{(2)}\mathbf{l}_0^{(2)})\right\}$$

$-2\times(4)\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

$$+\frac{1}{3}\text{vec}'\{\mathbf{J}_0^{(3)}-\mathrm{E}_g(\mathbf{J}_0^{(3)})\}\left(\mathbf{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)^{<3>}$$

$$\left.-\frac{1}{12}\text{vec}'\{\mathrm{E}_g(\mathbf{J}_0^{(4)})\}\left(\mathbf{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)^{<4>}\right]_{(B)O_p(n^{-2})}+O_p(n^{-5/2})$$

$$=-2(\overline{l}_0^*)_{O(1)}-2(\overline{l}_0-\overline{l}_0^*)_{O_p(n^{-1/2})}+\left(\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0'}\mathbf{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)_{O_p(n^{-1})}$$

$$+\left[-\text{vec}'(\mathbf{M})\left(\mathbf{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)^{<2>}+\frac{1}{3}\text{vec}'\{\mathrm{E}_g(\mathbf{J}_0^{(3)})\}\left(\mathbf{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)^{<3>}\right]_{O_p(n^{-3/2})}$$

$$-(n^{-2}\mathbf{q}_0^{*\prime}\mathbf{\Lambda}^{-1}\mathbf{q}_0^*)_{O(n^{-2})}$$

$$+\left[\text{vec}'(\mathbf{\Lambda})(\mathbf{\Lambda}^{(2)}\mathbf{l}_0^{(2)})^{<2>}+\frac{1}{3}\text{vec}'\{\mathbf{J}_0^{(3)}-\mathrm{E}_g(\mathbf{J}_0^{(3)})\}\left(\mathbf{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)^{<3>}\right.$$

$$\left.-\frac{1}{12}\text{vec}'\{\mathrm{E}_g(\mathbf{J}_0^{(4)})\}\left(\mathbf{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)^{<4>}\right]_{O_p(n^{-2})}+O_p(n^{-5/2})$$

$$\equiv-2(\overline{l}_0^*)_{O(1)}+\sum_{j=1}^{4}(\overline{l}_{\text{ML}}^{(j)})_{O_p(n^{-j/2})}-(n^{-2}\mathbf{q}_0^{*\prime}\mathbf{\Lambda}^{-1}\mathbf{q}_0^*)_{O(n^{-2})}+O_p(n^{-5/2})$$

$$(\overline{l}_{\text{W}}^{(j)}=\overline{l}_{\text{ML}}^{(j)},\ j=1,...,4),$$

where the underline with a number in parentheses indicates a quantity and the negative number e.g., "$-a \times (4)...$" indicates $-a$ times the quantity which has the symbol "(4)…" when the quantities with "$-a \times (4)...$" are summed.

In the last result of (A8.2), the first term for $\overline{l}_{\mathrm{W}}^{(4)}$ can also be written as

$$\mathrm{vec}'(\boldsymbol{\Lambda})(\boldsymbol{\Lambda}^{(2)}\mathbf{l}_0^{(2)})^{<2>}$$

$$= \mathrm{vec}'(\mathbf{M})\left\{\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)\otimes\left(\boldsymbol{\Lambda}^{-1}\mathbf{M}\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)\right\}$$

$$- \mathrm{vec}'\{\mathrm{E}_g(\mathbf{J}_0^{(3)})\}\left\{\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)^{<2>}\otimes\left(\boldsymbol{\Lambda}^{-1}\mathbf{M}\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)\right\} \qquad (A8.3)$$

$$+ \frac{1}{4}\mathrm{vec}'\{\mathrm{E}_g(\mathbf{J}_0^{(3)})\}\left[\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)^{<2>}\otimes\left\{\boldsymbol{\Lambda}^{-1}\mathrm{E}_g(\mathbf{J}_0^{(3)})\left(\boldsymbol{\Lambda}^{-1}\frac{\partial\overline{l}}{\partial\boldsymbol{\theta}_0}\right)^{<2>}\right\}\right]$$

(recall (A1.2)).

## A9. The derivation and actual expressions of Theorem 4

$$\kappa_{g1}(n^{-1}\mathrm{AIC_W})$$

$$= -2(\overline{l}_0^*)_{O(1)} + n^{-1}\{n\mathrm{E}_g(\overline{l}_{\mathrm{ML}}^{(2)}) + 2q\}_{O(1)}$$

$$+ n^{-2}\{n^2\mathrm{E}_g(\overline{l}_{\mathrm{ML}}^{(3)} + \overline{l}_{\mathrm{ML}}^{(4)}) - \mathbf{q}_0^{*'}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*\}_{O(1)} + O(n^{-3})$$

$$= -2\overline{l}_0^* + n^{-1}\{\mathrm{tr}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}) + 2q\}$$

$$+ n^{-2}\{n^2\mathrm{E}_g(\overline{l}_{\mathrm{ML}}^{(3)} + \overline{l}_{\mathrm{ML}}^{(4)}) - \mathbf{q}_0^{*'}\boldsymbol{\Lambda}^{-1}\mathbf{q}_0^*\} + O(n^{-3}) \qquad (A9.1)$$

$$\equiv -2\overline{l}_0^* + n^{-1}\alpha_{\mathrm{ML1}}^{(A)} + n^{-2}\alpha_{\mathrm{W\Delta1}}^{(A)} + O(n^{-3})$$

$$(\alpha_{\mathrm{W1}}^{(A)} = \alpha_{\mathrm{ML1}}^{(A)} = \mathrm{tr}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}) + 2q),$$

$$\kappa_{g2}(n^{-1}\mathrm{AIC_W}) = n^{-1}[n\mathrm{E}_g\{(\overline{l}_{\mathrm{ML}}^{(1)})^2\}]_{O(1)}$$

$$+ n^{-2}[2n^2\mathrm{E}_g(\overline{l}_{\mathrm{ML}}^{(1)}\overline{l}_{\mathrm{ML}}^{(2)}) + 2n^2\mathrm{E}_g(\overline{l}_{\mathrm{ML}}^{(1)}\overline{l}_{\mathrm{ML}}^{(3)})$$

$$+ n^2\mathrm{E}_g\{(\overline{l}_{\mathrm{ML}}^{(2)})^2\} - (\alpha_{\mathrm{ML1}}^{(A)} - 2q)^2] + O(n^{-3})$$

$$\equiv n^{-1}\alpha^{(A)}_{ML2} + n^{-2}\alpha^{(A)}_{ML\Delta2} + O(n^{-3})$$

$$(n\mathrm{E}_g(\overline{l}^{(2)}_{ML}) = \alpha^{(A)}_{ML1} - 2q = \mathrm{tr}(\boldsymbol{\Lambda}^{-1}\boldsymbol{\Gamma}))$$

$$(\alpha^{(A)}_{W2} = \alpha^{(A)}_{ML2} = n\mathrm{E}_g\{(\overline{l}^{(1)}_{ML})^2\} = 4\mathrm{E}_g\{(l_j - \overline{l}^*_0)^2\}$$

$$= 4\,\mathrm{var}_g(l_j),\ \alpha^{(A)}_{W\Delta2} = \alpha^{(A)}_{ML\Delta2}),$$

$$\kappa_{g3}(n^{-1}\mathrm{AIC}_W) = n^{-2}[\ n^2\mathrm{E}_g\{(\overline{l}^{(1)}_{ML})^3\} + 3n^2\mathrm{E}_g\{(\overline{l}^{(1)}_{ML})^2\,\overline{l}^{(2)}_{ML}\}$$

$$- 3n\mathrm{E}_g(\overline{l}^{(2)}_{ML})\alpha^{(A)}_{ML2}\ ] + O(n^{-3})$$

$$\equiv n^{-2}\alpha^{(A)}_{ML3} + O(n^{-3})\ (\alpha^{(A)}_{W3} = \alpha^{(A)}_{ML3}),$$

$$\kappa_{g4}(n^{-1}\mathrm{AIC}_W)$$

$$= \mathrm{E}_g[\{n^{-1}\mathrm{AIC}_W - \mathrm{E}_g(n^{-1}\mathrm{AIC}_W)\}^4]$$

$$- 3\{n^{-1}\alpha^{(A)}_{ML2} + n^{-2}\alpha^{(A)}_{ML\Delta2}\}^2 + O(n^{-4})$$

$$= \mathrm{E}_g\{(n^{-1}\mathrm{AIC}_W + 2\overline{l}^*_0)^4\}$$

$$+ n^{-3}[-4(\alpha^{(A)}_{ML1} - 2q)\{\alpha^{(A)}_{ML3} + 3(\alpha^{(A)}_{ML1} - 2q)\alpha^{(A)}_{ML2}\}$$

$$+ 6(\alpha^{(A)}_{ML1} - 2q)^2\alpha^{(A)}_{ML2}] - 3n^{-2}(\alpha^{(A)}_{ML2})^2$$

$$- 6n^{-3}\alpha^{(A)}_{ML2}\alpha^{(A)}_{ML\Delta2} + O(n^{-4})$$

$$= \mathrm{E}_g\{(n^{-1}\mathrm{AIC}_W + 2\overline{l}^*_0)^4\} - 3n^{-2}(\alpha^{(A)}_{ML2})^2$$

$$- n^{-3}\{4(\alpha^{(A)}_{ML1} - 2q)\alpha^{(A)}_{ML3} + 6\alpha^{(A)}_{ML2}\alpha^{(A)}_{ML\Delta2}$$

$$+ 6\alpha^{(A)}_{ML2}(\alpha^{(A)}_{ML1} - 2q)^2\} + O(n^{-4})$$

$$= n^{-3}[\ n^3\{\kappa_{g4}(\overline{l}^{(1)}_{ML})\}_{O(n^{-3})} + 4n^3\mathrm{E}_g\{(\overline{l}^{(1)}_{ML})^3\,\overline{l}^{(2)}_{ML}\}$$

$$+ 6n^3\mathrm{E}_g\{(\overline{l}^{(1)}_{ML})^2(\overline{l}^{(2)}_{ML})^2\}$$

$$+ 4n^3\mathrm{E}_g\{(\overline{l}^{(1)}_{ML})^3\,\overline{l}^{(3)}_{ML}\} - 4(\alpha^{(A)}_{ML1} - 2q)\alpha^{(A)}_{ML3}$$

$$- 6\alpha^{(A)}_{ML2}\alpha^{(A)}_{ML\Delta2} - 6\alpha^{(A)}_{ML2}(\alpha^{(A)}_{ML1} - 2q)^2\ ] + O(n^{-4})$$

$$\equiv n^{-3}\alpha^{(A)}_{ML4} + O(n^{-4})\ (\alpha^{(A)}_{W4} = \alpha^{(A)}_{ML4}).$$

## B. Expository notes in regression models

For $n^{-1}\mathrm{CAIC}$ under canonical parametrization, from Corollary 2,

$$c_1 = -\mathrm{vec}\,'(\mathbf{I}_0^{-1})\mathbf{J}_0^{(3)}\,'\mathbf{I}_0^{-1}\mathbf{J}_0^{(3)}\mathrm{vec}\,'(\mathbf{I}_0^{-1}) - \mathrm{vec}\,'(\mathbf{J}_0^{(3)})(\mathbf{I}_0^{-1})^{<3>}\mathrm{vec}(\mathbf{J}_0^{(3)})$$

$$- \mathrm{vec}\,'(\mathbf{J}_0^{(4)})\mathrm{vec}\{(\mathbf{I}_0^{-1})^{<2>}\}$$

$$= -\sum_{\substack{a,b,c,\\d,e,f=1}}^{p} i_0^{ab}(\mathbf{J}_0^{(3)})_{(a,b,c)}i_0^{cd}(\mathbf{J}_0^{(3)})_{(d,e,f)}i_0^{ef} - \sum_{\substack{a,b,c,\\d,e,f=1}}^{p} (\mathbf{J}_0^{(3)})_{(a,b,c)}i_0^{ad}i_0^{be}i_0^{cf}(\mathbf{J}_0^{(3)})_{(d,e,f)}$$

$$- \sum_{a,b,c,d=1}^{p} (\mathbf{J}_0^{(4)})_{(a,b,c,d)}i_0^{ab}i_0^{cd},$$

$$(\mathbf{I}_0^{-1})_{ab} = i_0^{ab}\ (a,b=1,...,p).$$

When the Jeffreys prior is used,

$$\mathbf{q}_0^* = \frac{1}{2}\frac{\partial\,\mathrm{vec}\,'(\mathbf{I})}{\partial\boldsymbol{\theta}_0}\mathrm{vec}(\mathbf{I}_0^{-1}),\quad \frac{\partial(\mathbf{I})_{ij}}{\partial\theta_{0k}} = -(\mathbf{J}_0^{(3)})_{(i,j,k)},$$

$$\frac{\partial(\mathbf{q}_0^*)_k}{\partial\theta_{0l*}} = \frac{1}{2}\frac{\partial^2\,\mathrm{vec}\,'(\mathbf{I})}{\partial\theta_{0k}\partial\theta_{0l*}}\mathrm{vec}(\mathbf{I}_0^{-1}) + \frac{1}{2}\frac{\partial\,\mathrm{vec}\,'(\mathbf{I})}{\partial\theta_{0k}}\mathrm{vec}\left(\frac{\partial\mathbf{I}^{-1}}{\partial\theta_{0l*}}\right),$$

$$\frac{\partial^2(\mathbf{I})_{ij}}{\partial\theta_{0k}\partial\theta_{0l*}} = -(\mathbf{J}_0^{(4)})_{(i,j,k,l*)},\quad \frac{\partial\mathbf{I}^{-1}}{\partial\theta_{0l*}} = -\mathbf{I}^{-1}\frac{\partial\mathbf{I}}{\partial\theta_{0l*}}\mathbf{I}^{-1}$$

$$(i,j,k,l^* = 1,...,p).$$

When the shape parameter in a regression model is unknown, $p+1$ in place of $p$ in $c_1$ and $\mathbf{q}_0^*$ should be used.

## B1. Logistic regression

$$\Pr(y_i^* = y_i \mid \pi_{0i}) = \pi_{0i}^{y_i}(1-\pi_{0i})^{1-y_i}\ (y_i = 0,1),$$

$$\pi_{0i} = \frac{1}{1+\exp(-\mathbf{x}_i\,'\boldsymbol{\beta}_0)},\quad \boldsymbol{\theta}_0 = \boldsymbol{\beta}_0,\ \mathbf{X} = (\mathbf{x}_1,...,\mathbf{x}_n)\,',\ \mathbf{y} = (y_1,...,y_n)\,',$$

$$\pi_i = \frac{1}{1+\exp(-\mathbf{x}_i\,'\boldsymbol{\beta})},\quad \boldsymbol{\theta} = \boldsymbol{\beta}\ (i=1,...,n),$$

$$\bar{l}(\boldsymbol{\theta} \mid \mathbf{X}, \mathbf{y}) = \bar{l}(\boldsymbol{\theta}) \equiv \bar{l} = n^{-1} \sum_{i=1}^{n} \{y_i \log \pi_i + (1 - y_i) \log(1 - \pi_i)\}$$

$$= n^{-1} \sum_{i=1}^{n} \{y_i \mathbf{x}_i' \boldsymbol{\beta} + \log(1 - \pi_i)\},$$

$$\frac{\partial \bar{l}}{\partial \boldsymbol{\beta}} = n^{-1} \sum_{i=1}^{n} (y_i - \pi_i) \mathbf{x}_i, \quad \hat{\boldsymbol{\beta}}_{\mathrm{ML}} \quad \text{is given by solving} \quad \frac{\partial \bar{l}}{\partial \boldsymbol{\beta}} \Big|_{\boldsymbol{\beta} = \hat{\boldsymbol{\beta}}_{\mathrm{ML}}} = \mathbf{0},$$

$$\frac{\partial \bar{l}}{\partial \boldsymbol{\beta}_0} = n^{-1} \sum_{i=1}^{n} (y_i - \pi_{0i}) \mathbf{x}_i,$$

$$\boldsymbol{\Lambda} = \frac{\partial^2 \bar{l}}{\partial \boldsymbol{\beta}_0 \partial \boldsymbol{\beta}_0'} = -n^{-1} \sum_{i=1}^{n} \pi_{0i}(1 - \pi_{0i}) \mathbf{x}_i \mathbf{x}_i' = -n^{-1} \sum_{i=1}^{n} \kappa_2(y_i^*) \mathbf{x}_i \mathbf{x}_i' = -\mathbf{I}_0,$$

$$\boldsymbol{\Gamma} = n \mathrm{E}_f \left( \frac{\partial \bar{l}}{\partial \boldsymbol{\beta}_0} \frac{\partial \bar{l}}{\partial \boldsymbol{\beta}_0'} \right) = n^{-1} \sum_{i=1}^{n} \mathrm{E}_f \{(y_i^* - \pi_{0i})^2\} \mathbf{x}_i \mathbf{x}_i'$$

$$= n^{-1} \sum_{i=1}^{n} \pi_{0i}(1 - \pi_{0i}) \mathbf{x}_i \mathbf{x}_i' = \mathbf{I}_0 \ (\mathrm{var}(y_i^*) = \pi_{0i}(1 - \pi_{0i})),$$

$$(\mathbf{J}_0^{(3)})_{(a,b,c)} = \frac{\partial^3 \bar{l}}{\partial \beta_{0a} \partial \beta_{0b} \partial \beta_{0c}} = -n^{-1} \sum_{i=1}^{n} (1 - 2\pi_{0i}) \pi_{0i}(1 - \pi_{0i}) x_{ia} x_{ib} x_{ic}$$

$$= -n^{-1} \sum_{i=1}^{n} \kappa_3(y_i^*) x_{ia} x_{ib} x_{ic},$$

$$(\mathbf{J}_0^{(4)})_{(a,b,c,d)} = \frac{\partial^4 \bar{l}}{\partial \beta_{0a} \partial \beta_{0b} \partial \beta_{0c} \partial \beta_{0d}}$$

$$= -n^{-1} \sum_{i=1}^{n} (1 - 6\pi_{0i} + 6\pi_{0i}^2) \pi_{0i}(1 - \pi_{0i}) x_{ia} x_{ib} x_{ic} x_{id}$$

$$= -n^{-1} \sum_{i=1}^{n} \kappa_4(y_i^*) x_{ia} x_{ib} x_{ic} x_{id}$$

$$(a, b, c, d = 1, ..., n),$$

$$-2\hat{\bar{l}}_{\mathrm{ML}} = -2n^{-1}\sum_{i=1}^{n}\{y_i \log \hat{\pi}_{\mathrm{ML}i} + (1-y_i)\log(1-\hat{\pi}_{\mathrm{ML}i})\}$$

$$= -2n^{-1}\sum_{i=1}^{n}\{y_i \mathbf{x}_i {}'\hat{\boldsymbol{\beta}}_{\mathrm{ML}} + \log(1-\hat{\pi}_{\mathrm{ML}i})\},$$

$$\hat{\pi}_{\mathrm{ML}i} = \frac{1}{1+\exp(-\mathbf{x}_i{}'\hat{\boldsymbol{\beta}}_{\mathrm{ML}})},$$

$$n^{-1}\mathrm{AIC} = -2\hat{\bar{l}}_{\mathrm{ML}} + n^{-1}2p,$$

$$n^{-1}\mathrm{CAIC} = -2\hat{\bar{l}}_{\mathrm{ML}} + n^{-1}2p - n^{-2}\hat{c}_1.$$

## B2. Poisson regression

$$\Pr(y_i^* = y_i \mid \lambda_{0i}) = \lambda_{0i}^{y_i}\exp(-\lambda_{0i})/y_i!\ (y_i = 0,1,2,...),$$

$$\lambda_{0i} = \exp(\mathbf{x}_i{}'\boldsymbol{\beta}_0),\ \boldsymbol{\theta}_0 = \boldsymbol{\beta}_0,\ \lambda_i = \exp(\mathbf{x}_i{}'\boldsymbol{\beta}),\ \boldsymbol{\theta} = \boldsymbol{\beta}\ (i = 1,...,n),$$

$$\bar{l} = n^{-1}\sum_{i=1}^{n}\{y_i\mathbf{x}_i{}'\boldsymbol{\beta} - \lambda_i - \log(y_i!)\},$$

$$\frac{\partial \bar{l}}{\partial \boldsymbol{\beta}} = n^{-1}\sum_{i=1}^{n}(y_i - \lambda_i)\mathbf{x}_i,\quad \hat{\boldsymbol{\beta}}_{\mathrm{ML}}\ \text{ is given by solving }\ \frac{\partial \bar{l}}{\partial \boldsymbol{\beta}}\Big|_{\boldsymbol{\beta}=\hat{\boldsymbol{\beta}}_{\mathrm{ML}}} = \mathbf{0},$$

$$\frac{\partial \bar{l}}{\partial \boldsymbol{\beta}_0} = n^{-1}\sum_{i=1}^{n}(y_i - \lambda_{0i})\mathbf{x}_i\ (\mathrm{E}_f(y_i^*) = \lambda_{0i}),$$

$$\boldsymbol{\Lambda} = \frac{\partial^2 \bar{l}}{\partial \boldsymbol{\beta}_0 \partial \boldsymbol{\beta}_0{}'} = -n^{-1}\sum_{i=1}^{n}\lambda_{0i}\,\mathbf{x}_i\mathbf{x}_i{}' = -n^{-1}\sum_{i=1}^{n}\kappa_2(y_i^*)\mathbf{x}_i\mathbf{x}_i{}' = -\mathbf{I}_0,$$

$$\boldsymbol{\Gamma} = n\mathrm{E}_f\left(\frac{\partial \bar{l}}{\partial \boldsymbol{\beta}_0}\frac{\partial \bar{l}}{\partial \boldsymbol{\beta}_0{}'}\right) = n^{-1}\sum_{i=1}^{n}\mathrm{E}_f\{(y_i^* - \lambda_{0i})^2\}\mathbf{x}_i\mathbf{x}_i{}' = n^{-1}\sum_{i=1}^{n}\lambda_{0i}\mathbf{x}_i\mathbf{x}_i{}' = \mathbf{I}_0$$

$$(\mathrm{var}(y_i^*) = \lambda_{0i}),$$

$$(\mathbf{J}_0^{(3)})_{(a,b,c)} = \frac{\partial^3 \overline{l}}{\partial \beta_{0a} \partial \beta_{0b} \partial \beta_{0c}} = -n^{-1} \sum_{i=1}^{n} \lambda_{0i} \, x_{ia} x_{ib} x_{ic}$$

$$= -n^{-1} \sum_{i=1}^{n} \kappa_3(y_i^*) x_{ia} x_{ib} x_{ic},$$

$$(\mathbf{J}_0^{(4)})_{(a,b,c,d)} = \frac{\partial^4 \overline{l}}{\partial \beta_{0a} \partial \beta_{0b} \partial \beta_{0c} \partial \beta_{0d}} = -n^{-1} \sum_{i=1}^{n} \lambda_{0i} \, x_{ia} x_{ib} x_{ic} x_{id}$$

$$= -n^{-1} \sum_{i=1}^{n} \kappa_4(y_i^*) x_{ia} x_{ib} x_{ic} x_{id}$$

$$(a,b,c,d = 1,...,n),$$

$$-2\hat{\overline{l}}_{\mathrm{ML}} = -2n^{-1} \sum_{i=1}^{n} \{y_i \mathbf{x}_i \,'\hat{\boldsymbol{\beta}}_{\mathrm{ML}} - \hat{\lambda}_{\mathrm{ML}i} - \log(y_i!)\}, \quad \hat{\lambda}_{\mathrm{ML}i} = \exp(\mathbf{x}_i \,'\hat{\boldsymbol{\beta}}_{\mathrm{ML}}),$$

$$n^{-1}\mathrm{AIC} = -2\hat{\overline{l}}_{\mathrm{ML}} + n^{-1} 2p,$$

$$n^{-1}\mathrm{CAIC} = -2\hat{\overline{l}}_{\mathrm{ML}} + n^{-1} 2p - n^{-2}\hat{c}_1.$$

## B3. Negative binomial (NB) regression
## B3.1 NB regression when the shape parameter $r$ is known

$$\Pr(y_i^* = y_i \mid \pi_{0i}, r) = \binom{y_i + r - 1}{y_i} \pi_{0i}^{y_i} (1 - \pi_{0i})^r$$

$$= \frac{\Gamma(y_i + r)}{y_i! \Gamma(r)} \pi_{0i}^{y_i} (1 - \pi_{0i})^r \, (y_i = 0,1,2,...),$$

$$\pi_{0i} = \exp(\mathbf{x}_i \,'\boldsymbol{\beta}_0), \; \boldsymbol{\theta}_0 = \boldsymbol{\beta}_0, \; \pi_i = \exp(\mathbf{x}_i \,'\boldsymbol{\beta}), \; \boldsymbol{\theta} = \boldsymbol{\beta} \; (i = 1,...,n),$$

$$\overline{l} = n^{-1} \sum_{i=1}^{n} \{\log \Gamma(y_i + r) - \log(y_i!) - \log \Gamma(r) + y_i \mathbf{x}_i \,'\boldsymbol{\beta} + r \log(1 - \pi_i)\},$$

$$\frac{\partial \overline{l}}{\partial \boldsymbol{\beta}} = n^{-1} \sum_{i=1}^{n} \left( y_i - \frac{r\pi_i}{1 - \pi_i} \right) \mathbf{x}_i, \quad \hat{\boldsymbol{\beta}}_{\mathrm{ML}} \text{ is given by solving } \frac{\partial \overline{l}}{\partial \boldsymbol{\beta}} \Big|_{\boldsymbol{\beta} = \hat{\boldsymbol{\beta}}_{\mathrm{ML}}} = \mathbf{0},$$

$$\frac{\partial \overline{l}}{\partial \boldsymbol{\beta}_0} = n^{-1} \sum_{i=1}^{n} \left( y_i - \frac{r\pi_{0i}}{1 - \pi_{0i}} \right) \mathbf{x}_i \quad (\mathrm{E}_f(y_i^*) = r\pi_{0i} / (1 - \pi_{0i})),$$

$$\boldsymbol{\Lambda} = \frac{\partial^2 \overline{l}}{\partial \boldsymbol{\beta}_0 \partial \boldsymbol{\beta}_0{}'} = -n^{-1} \sum_{i=1}^{n} \frac{r\pi_{0i}}{(1 - \pi_{0i})^2} \mathbf{x}_i \mathbf{x}_i{}' = -n^{-1} \sum_{i=1}^{n} \kappa_2(y_i^*) \mathbf{x}_i \mathbf{x}_i{}' = -\mathbf{I}_0,$$

$$\boldsymbol{\Gamma} = n\mathrm{E}_f \left( \frac{\partial \overline{l}}{\partial \boldsymbol{\beta}_0} \frac{\partial \overline{l}}{\partial \boldsymbol{\beta}_0{}'} \right) = n^{-1} \sum_{i=1}^{n} \mathrm{E}_f \left\{ \left( y_i^* - \frac{r\pi_{0i}}{1 - \pi_{0i}} \right)^2 \right\} \mathbf{x}_i \mathbf{x}_i{}'$$

$$= n^{-1} \sum_{i=1}^{n} \frac{r\pi_{0i}}{(1 - \pi_{0i})^2} \mathbf{x}_i \mathbf{x}_i{}' = \mathbf{I}_0 \quad (\mathrm{var}(y_i^*) = r\pi_{0i} / (1 - \pi_{0i})^2),$$

$$(\mathbf{J}_0^{(3)})_{(a,b,c)} = \frac{\partial^3 \overline{l}}{\partial \beta_{0a} \partial \beta_{0b} \partial \beta_{0c}}$$

$$= -n^{-1} \sum_{i=1}^{n} r\pi_{0i} \left\{ \frac{1}{(1 - \pi_{0i})^2} + \frac{2\pi_{0i}}{(1 - \pi_{0i})^3} \right\} x_{ia} x_{ib} x_{ic}$$

$$= -n^{-1} \sum_{i=1}^{n} \kappa_3(y_i^*) x_{ia} x_{ib} x_{ic},$$

$$(\mathbf{J}_0^{(4)})_{(a,b,c,d)} = \frac{\partial^4 \overline{l}}{\partial \beta_{0a} \partial \beta_{0b} \partial \beta_{0c} \partial \beta_{0d}}$$

$$= -n^{-1} \sum_{i=1}^{n} r\pi_{0i} \left\{ \frac{1}{(1 - \pi_{0i})^2} + \frac{6\pi_{0i}}{(1 - \pi_{0i})^3} + \frac{6\pi_{0i}^2}{(1 - \pi_{0i})^4} \right\} x_{ia} x_{ib} x_{ic} x_{id}$$

$$= -n^{-1} \sum_{i=1}^{n} \kappa_4(y_i^*) x_{ia} x_{ib} x_{ic} x_{id}$$

$$(a, b, c, d = 1, ..., n),$$

$$-2\hat{\overline{l}}_{\mathrm{ML}} = -2n^{-1} \sum_{i=1}^{n} \{ \log \Gamma(y_i + r) - \log(y_i!) - \log \Gamma(r)$$

$$+ y_i \mathbf{x}_i{}' \hat{\boldsymbol{\beta}}_{\mathrm{ML}} + r \log(1 - \hat{\pi}_{\mathrm{ML}i}),$$

$$\hat{\pi}_{\mathrm{ML}i} = \exp(\mathbf{x}_i{}'\hat{\boldsymbol{\beta}}_{\mathrm{ML}}),$$

$$n^{-1}\mathrm{AIC} = -2\hat{\bar{l}}_{\mathrm{ML}} + n^{-1}2p,$$

$$n^{-1}\mathrm{CAIC} = -2\hat{\bar{l}}_{\mathrm{ML}} + n^{-1}2p - n^{-2}\hat{c}_1.$$

**B3.2 NB regression when the shape parameter $r_0$ is unknown**

$$\boldsymbol{\theta}_0 = (\boldsymbol{\beta}_0{}',\, r_0)', \quad \boldsymbol{\theta} = (\boldsymbol{\beta}',\, r)',$$

$$\bar{l} = n^{-1}\sum_{i=1}^{n}\{\log\Gamma(y_i+r) - \log(y_i!) - \log\Gamma(r) + y_i\mathbf{x}_i{}'\boldsymbol{\beta} + r\log(1-\pi_i)\},$$

$$\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}} = \begin{pmatrix} n^{-1}\sum_{i=1}^{n}\left(y_i - \dfrac{r\pi_i}{1-\pi_i}\right)\mathbf{x}_i \\[2ex] n^{-1}\sum_{i=1}^{n}\{\psi(y_i+r) - \psi(r) + \log(1-\pi_i)\} \end{pmatrix},$$

$\hat{\boldsymbol{\beta}}_{\mathrm{ML}}$ is given by solving $\left.\dfrac{\partial \bar{l}}{\partial \boldsymbol{\beta}}\right|_{\boldsymbol{\beta}=\hat{\boldsymbol{\beta}}_{\mathrm{ML}}} = \mathbf{0}$,

$$\frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} = \begin{pmatrix} n^{-1}\sum_{i=1}^{n}\left(y_i - \dfrac{r_0\pi_{0i}}{1-\pi_{0i}}\right)\mathbf{x}_i \\[2ex] n^{-1}\sum_{i=1}^{n}\{\psi(y_i+r_0) - \psi(r_0) + \log(1-\pi_{0i})\} \end{pmatrix}$$

$$(\mathrm{E}_f(y_i^*) = r_0\pi_{0i}/(1-\pi_{0i})),$$

$$\boldsymbol{\Lambda} = \frac{\partial^2 \bar{l}}{\partial \boldsymbol{\theta}_0 \partial \boldsymbol{\theta}_0{}'} = -\begin{pmatrix} n^{-1}\sum_{i=1}^{n}\dfrac{r_0\pi_{0i}}{(1-\pi_{0i})^2}\mathbf{x}_i\mathbf{x}_i{}' & n^{-1}\sum_{i=1}^{n}\dfrac{\pi_{0i}}{1-\pi_{0i}}\mathbf{x}_i \\[2ex] n^{-1}\sum_{i=1}^{n}\dfrac{\pi_{0i}}{1-\pi_{0i}}\mathbf{x}_i{}' & -n^{-1}\sum_{i=1}^{n}\{\psi'(y_i+r_0) - \psi'(r_0)\} \end{pmatrix}$$

$$\neq -\mathbf{I}_0$$

(note that the $(p+1, p+1)$th element of $\boldsymbol{\Lambda}$ is stochastic).

Non-zero elements of $\mathbf{J}_0^{(3)}$ and $\mathbf{J}_0^{(4)}$ are

$$(\mathbf{J}_0^{(3)})_{(a,b,c)} = \frac{\partial^3 \overline{l}}{\partial \beta_{0a} \partial \beta_{0b} \partial \beta_{0c}}$$

$$= -n^{-1} \sum_{i=1}^{n} r_0 \pi_{0i} \left\{ \frac{1}{(1-\pi_{0i})^2} + \frac{2\pi_{0i}}{(1-\pi_{0i})^3} \right\} x_{ia} x_{ib} x_{ic},$$

$$(\mathbf{J}_0^{(3)})_{(a,b,p+1)} = \frac{\partial^3 \overline{l}}{\partial \beta_{0a} \partial \beta_{0b} \partial r_0} = -n^{-1} \sum_{i=1}^{n} \frac{\pi_{0i}}{(1-\pi_{0i})^2} x_{ia} x_{ib}$$

$$(a,b,c=1,...,p),$$

$$(\mathbf{J}_0^{(3)})_{(p+1,p+1,p+1)} = \frac{\partial^3 \overline{l}}{\partial r_0^3} = n^{-1} \sum_{i=1}^{n} \{\psi''(y_i + r_0) - \psi''(r_0)\}$$

(note that the last element is stochastic),

$$(\mathbf{J}_0^{(4)})_{(a,b,c,d)} = \frac{\partial^4 \overline{l}}{\partial \beta_{0a} \partial \beta_{0b} \partial \beta_{0c} \partial \beta_{0d}}$$

$$= -n^{-1} \sum_{i=1}^{n} r_0 \pi_{0i} \left\{ \frac{1}{(1-\pi_{0i})^2} + \frac{6\pi_{0i}}{(1-\pi_{0i})^3} + \frac{6\pi_{0i}^2}{(1-\pi_{0i})^4} \right\} x_{ia} x_{ib} x_{ic} x_{id}$$

$$(\mathbf{J}_0^{(4)})_{(a,b,c,p+1)} = \frac{\partial^4 \overline{l}}{\partial \beta_{0a} \partial \beta_{0b} \partial \beta_{0c} \partial r_0}$$

$$= -n^{-1} \sum_{i=1}^{n} \pi_{0i} \left\{ \frac{1}{(1-\pi_{0i})^2} + \frac{2\pi_{0i}}{(1-\pi_{0i})^3} \right\} x_{ia} x_{ib}$$

$$(a,b,c,d=1,...,p),$$

$$(\mathbf{J}_0^{(4)})_{(p+1,p+1,p+1,p+1)} = \frac{\partial^4 \overline{l}}{\partial r_0^4} = n^{-1} \sum_{i=1}^{n} \{\psi'''(y_i + r_0) - \psi'''(r_0)\}$$

(note that the last element is stochastic),

$$n^{-1}\text{AIC} = -2\hat{\overline{l}}_{\text{ML}} + n^{-1}2(p+1),$$

$$n^{-1}\text{CAIC}^* = -2\hat{\overline{l}}_{\text{ML}} + n^{-1}2(p+1) - n^{-2}\hat{c}_1 \quad (n^{-1}\text{CAIC}^* \doteq n^{-1}\text{CAIC}).$$

**B4. Gamma regression**
**B4.1 Gamma regression when the shape parameter $\alpha$ is known**

$$f(y_i^* = y_i \mid \lambda_{0i}, \alpha) = y_i^{\alpha-1} \lambda_{0i}^{\alpha} \exp(-\lambda_{0i} y_i) / \Gamma(\alpha) \ (y_i > 0),$$

$$\lambda_{0i} = \mathbf{x}_i' \boldsymbol{\beta}_0, \ \boldsymbol{\theta}_0 = \boldsymbol{\beta}_0, \ \lambda_i = \mathbf{x}_i' \boldsymbol{\beta}, \ \boldsymbol{\theta} = \boldsymbol{\beta} \ (i = 1,...,n),$$

$$\kappa_j(y_i^*) = (j-1)! \alpha / \lambda_{0i}^j \ (j = 1, 2,...),$$

$$\bar{l} = n^{-1} \sum_{i=1}^{n} \{(\alpha - 1)\log(y_i) + \alpha \log(\lambda_i) - \lambda_i y_i\} - \log \Gamma(\alpha),$$

$$\frac{\partial \bar{l}}{\partial \boldsymbol{\beta}} = n^{-1} \sum_{i=1}^{n} \left( \frac{\alpha}{\lambda_i} - y_i \right) \mathbf{x}_i, \quad \hat{\boldsymbol{\beta}}_{\mathrm{ML}} \text{ is given by solving } \left. \frac{\partial \bar{l}}{\partial \boldsymbol{\beta}} \right|_{\boldsymbol{\beta} = \hat{\boldsymbol{\beta}}_{\mathrm{ML}}} = \mathbf{0},$$

$$\frac{\partial \bar{l}}{\partial \boldsymbol{\beta}_0} = n^{-1} \sum_{i=1}^{n} \left( \frac{\alpha}{\lambda_{0i}} - y_i \right) \mathbf{x}_i \ (\mathrm{E}_f(y_i^*) = \alpha / \lambda_{0i}),$$

$$\boldsymbol{\Lambda} = \frac{\partial^2 \bar{l}}{\partial \boldsymbol{\beta}_0 \partial \boldsymbol{\beta}_0'} = -n^{-1} \alpha \sum_{i=1}^{n} \frac{\mathbf{x}_i \mathbf{x}_i'}{\lambda_{0i}^2} = -n^{-1} \sum_{i=1}^{n} \kappa_2(-y_i^*) \mathbf{x}_i \mathbf{x}_i'$$

$$= -n^{-1} \sum_{i=1}^{n} \kappa_2(y_i^*) \mathbf{x}_i \mathbf{x}_i' = -\mathbf{I}_0,$$

$$\boldsymbol{\Gamma} = n \mathrm{E}_f \left( \frac{\partial \bar{l}}{\partial \boldsymbol{\beta}_0} \frac{\partial \bar{l}}{\partial \boldsymbol{\beta}_0'} \right) = n^{-1} \sum_{i=1}^{n} \mathrm{E}_f \left\{ \left( y_i^* - \frac{\alpha}{\lambda_{0i}} \right)^2 \right\} \mathbf{x}_i \mathbf{x}_i'$$

$$= n^{-1} \alpha \sum_{i=1}^{n} \frac{\mathbf{x}_i \mathbf{x}_i'}{\lambda_{0i}^2} = \mathbf{I}_0 \quad (\mathrm{var}(y_i^*) = \alpha / \lambda_{0i}^2),$$

$$(\mathbf{J}_0^{(3)})_{(a,b,c)} = \frac{\partial^3 \bar{l}}{\partial \beta_{0a} \partial \beta_{0b} \partial \beta_{0c}} = n^{-1} 2\alpha \sum_{i=1}^{n} \frac{x_{ia} x_{ib} x_{ic}}{\lambda_{0i}^3},$$

$$= -n^{-1} \sum_{i=1}^{n} \kappa_3(-y_i^*) x_{ia} x_{ib} x_{ic} = n^{-1} \sum_{i=1}^{n} \kappa_3(y_i^*) x_{ia} x_{ib} x_{ic},$$

$$(\mathbf{J}_0^{(4)})_{(a,b,c,d)} = \frac{\partial^4 \overline{l}}{\partial \beta_{0a} \partial \beta_{0b} \partial \beta_{0c} \partial \beta_{0d}} = -n^{-1} 6\alpha \sum_{i=1}^{n} \frac{x_{ia} x_{ib} x_{ic} x_{id}}{\lambda_{0i}^3}$$

$$= -n^{-1} \sum_{i=1}^{n} \kappa_4(-y_i^*) x_{ia} x_{ib} x_{ic} x_{id} = -n^{-1} \sum_{i=1}^{n} \kappa_4(y_i^*) x_{ia} x_{ib} x_{ic} x_{id}$$

$$(a,b,c,d=1,...,n),$$

$$-2\hat{\overline{l}}_{\mathrm{ML}} = -2n^{-1} \sum_{i=1}^{n} \{(\alpha-1)\log(y_i) + \alpha \log(\hat{\lambda}_{\mathrm{ML}i}) - \hat{\lambda}_{\mathrm{ML}i} y_i\} + 2\log\Gamma(\alpha)$$

$$\hat{\lambda}_{\mathrm{ML}i} = \mathbf{x}_i' \hat{\boldsymbol{\beta}}_{\mathrm{ML}},$$

$$n^{-1}\mathrm{AIC} = -2\hat{\overline{l}}_{\mathrm{ML}} + n^{-1} 2p,$$

$$n^{-1}\mathrm{CAIC} = -2\hat{\overline{l}}_{\mathrm{ML}} + n^{-1} 2p - n^{-2}\hat{c}_1.$$

**B4.2 Gamma regression when the shape parameter $\alpha_0$ is unknown**

$$\boldsymbol{\theta}_0 = (\boldsymbol{\beta}_0', \alpha_0)', \quad \boldsymbol{\theta} = (\boldsymbol{\beta}', \alpha)',$$

$$\overline{l} = n^{-1} \sum_{i=1}^{n} \{(\alpha-1)\log(y_i) + \alpha \log(\lambda_i) - \lambda_i y_i\} - \log\Gamma(\alpha),$$

$$\frac{\partial \overline{l}}{\partial \boldsymbol{\theta}} = \begin{pmatrix} n^{-1} \sum_{i=1}^{n} \left( \dfrac{\alpha}{\lambda_i} - y_i \right) \mathbf{x}_i \\ n^{-1} \sum_{i=1}^{n} \log(\lambda_i y_i) - \psi(\alpha) \end{pmatrix},$$

$\hat{\boldsymbol{\beta}}_{\mathrm{ML}}$ is given by solving $\left. \dfrac{\partial \overline{l}}{\partial \boldsymbol{\beta}} \right|_{\boldsymbol{\beta}=\hat{\boldsymbol{\beta}}_{\mathrm{ML}}} = \mathbf{0}$,

$$\frac{\partial \overline{l}}{\partial \boldsymbol{\theta}_0} = \begin{pmatrix} n^{-1} \sum_{i=1}^{n} \left( \frac{\alpha_0}{\lambda_{0i}} - y_i \right) \mathbf{x}_i \\ n^{-1} \sum_{i=1}^{n} \log(\lambda_i y_i) - \psi(\alpha_0) \end{pmatrix} \quad (\mathrm{E}_f(y_i^*) = \alpha_0 / \lambda_{0i}),$$

$$\boldsymbol{\Lambda} = \frac{\partial^2 \overline{l}}{\partial \boldsymbol{\theta}_0 \partial \boldsymbol{\theta}_0'} = - \begin{pmatrix} n^{-1} \alpha_0 \sum_{i=1}^{n} \frac{\mathbf{x}_i \mathbf{x}_i'}{\lambda_{0i}^2} & -n^{-1} \sum_{i=1}^{n} \frac{\mathbf{x}_i}{\lambda_{0i}} \\ -n^{-1} \sum_{i=1}^{n} \frac{\mathbf{x}_i'}{\lambda_{0i}} & \psi'(\alpha_0) \end{pmatrix} = -\mathbf{I}_0.$$

Non-zero elements of $\mathbf{J}_0^{(3)}$ and $\mathbf{J}_0^{(4)}$ are

$$(\mathbf{J}_0^{(3)})_{(a,b,c)} = \frac{\partial^3 \overline{l}}{\partial \beta_{0a} \partial \beta_{0b} \partial \beta_{0c}} = n^{-1} 2\alpha_0 \sum_{i=1}^{n} \frac{x_{ia} x_{ib} x_{ic}}{\lambda_{0i}^3},$$

$$(\mathbf{J}_0^{(3)})_{(a,b,p+1)} = \frac{\partial^3 \overline{l}}{\partial \beta_{0a} \partial \beta_{0b} \partial \alpha_0} = -n^{-1} \sum_{i=1}^{n} \frac{x_{ia} x_{ib}}{\lambda_{0i}^2}$$

$$(a,b,c = 1,...,p),$$

$$(\mathbf{J}_0^{(3)})_{(p+1,p+1,p+1)} = \frac{\partial^3 \overline{l}}{\partial \alpha_0^3} = -\psi''(\alpha_0),$$

$$(\mathbf{J}_0^{(4)})_{(a,b,c,d)} = \frac{\partial^4 \overline{l}}{\partial \beta_{0a} \partial \beta_{0b} \partial \beta_{0c} \partial \beta_{0d}} = -n^{-1} 6\alpha_0 \sum_{i=1}^{n} \frac{x_{ia} x_{ib} x_{ic} x_{id}}{\lambda_{0i}^3}$$

$$(\mathbf{J}_0^{(4)})_{(a,b,c,p+1)} = \frac{\partial^3 \overline{l}}{\partial \beta_{0a} \partial \beta_{0b} \partial \beta_{0c} \partial \alpha_0} = n^{-1} 2 \sum_{i=1}^{n} \frac{x_{ia} x_{ib} x_{ic}}{\lambda_{0i}^3}$$

$$(a,b,c,d = 1,...,n),$$

$$(\mathbf{J}_0^{(4)})_{(p+1,p+1,p+1,p+1)} = \frac{\partial^4 \overline{l}}{\partial \alpha_0^4} = -\psi'''(\alpha_0),$$

$$n^{-1} \mathrm{AIC} = -2\hat{\overline{l}}_{\mathrm{ML}} + n^{-1} 2(p+1),$$

$$n^{-1} \mathrm{CAIC} = -2\hat{\overline{l}}_{\mathrm{ML}} + n^{-1} 2(p+1) - n^{-2} \hat{c}_1.$$

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