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No.0604

OUC Economic Research Papers

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September 6, 2006

Abstract

This paper investigates a long-run equilibrium of the Tullock contest using an evolutionary game-theoretic approach. The finite population evolutionarily stable strategy (ESS) yields overdissipation of the rent when there are increasing returns to expenditure. However, imitative behavior, which should be a source of the evolutionary dynamics behind the ESS, is implausible because individual rationality is not always satisfied. In this paper, we attempt to specify such implicit imitative behavior and construct explicit evolutionary dynamics. Under our plausible imitation rule, we will show that full dissipation may prevail in the long run as long as there are increasing returns.

***JEL Classification*:** D72, C73

***Keywords*:** Contests, Overdissipation, Imitative learning, Stochastically stable state

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*This research was partially supported by a Grant-in-Aid for Science Research (no.17730177) from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

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1 Introduction

The seminal work of Tullock (1980) has inspired numerous studies on rent-seeking contests. However, at the same time a problem emerged: The Nash equilibria may not exist in the rent-seeking game when there are excessive increasing returns to rent-seeking expenditure. In the static Tullock contest, the degree of marginal returns to rent-seeking expenditure is one of the most important factors for determining the rate of rent dissipation, the main area of interest in the rent-seeking theory. Tullock (1980) showed that the rate of rent dissipation is less than or equal to 1 when finite rent-seekers engage in rent-seeking activities provided that a parameter r, which determines the degree of marginal returns, is smaller than or equal to a threshold value. In the case that r is greater than the threshold value, however, the static Tullock rent-seeking game cannot establish a pure strategy Nash equilibrium. Therefore, we cannot confirm whether overdissipation of the rent occurs in the Tullock model.

If the rent-seeking game has finite players and a finite strategy space, we can verify that a mixed-strategy Nash equilibrium exists by a mixed extension of the strategy space. Baye, Kovenock, and de Vries (1994) confirmed that in the symmetric mixed-strategy Nash equilibrium of a finite rent-seeking game, overdissipation of the rent does not occur. Moreover, they showed that even in a rent-seeking game with a continuous strategy space, there exists a symmetric mixed-strategy Nash equilibrium that does not yield overdissipation of the rent.

Recently, Hehenkamp, Leininger, and Possajennikov (2004) adopted the concept of a finite population evolutionarily stable strategy (ESS), presented by Schaffer (1988), preserving the basic framework of the original Tullock contest. They show that there exists a finite population ESS, and the ESS entails overdissipation of the rent when r is excessively high. However, the solution using the concept of an ESS involves a drawback because it is not possible to identify dynamic forces in order to have players adopt evolutionarily stable behaviors. Since evolutionary forces are produced through interactions among boundedly rational rent-seekers, by revealing the source of the forces, we can determine how the rationality of the rent-seekers should be bounded. Therefore, we cannot conclude that overdissipation will be present in real life contests if the ESS is supported by implausibly irrational behavior.

Hehenkamp et al. (2004) state that imitative behavior among rent-seekers is obviously the source of the evolutionary force behind the finite population ESS. In this paper, therefore, we attempt to specify such implicit imitative behavior and construct explicit evolutionary dynamics. We implement discrete-time dynamics and assume that boundedly rational rent-seekers imitate one of the currently most successful strategies in the next period. The purpose of this paper is to investigate long-run equilibrium as a consequence of equilibrium selection in evolutionary dynamics and to consider whether the equilibrium strategies are reasonable from the standpoint of reality.

For this purpose, we adopt the evolutionary equilibrium concept of *stochastic stability* instead of evolutionary stability. We demonstrate that in our imitative learning dynamics, there exists a unique stochastically stable state (SSS) and that the long-run equilibrium strategy profile constituting the unique SSS coincides with the profile of ESSs, as long as an ESS exists. Hence, overdissipation of the rent arises in the long-run equilibrium of the SSS.

We confirm that the only factor to cause overdissipation in the SSS is the excessively irrational behavior of the rent-seekers as they imitate the most successful strategy even though it yields a negative payoff. We consider a modified imitation rule in which such an implausibly irrational behavior is excluded and demonstrate that the rent is almost always fully dissipated in the long run when there are increasing returns to rent-seeking expenditure.

The remainder of this paper is organized as follows. Section 2 reviews the static Tullock contest. In Section 3, we derive a stochastically stable state in imitative learning dynamics and compare the results with the finite population ESS. Section 4 defines a modified imitation rule and determines a stochastically stable state under the rule; Section 5 discusses the results under the rule and concludes the paper.

2 The static Tullock contest

Consider a contest in which N (≥ 2) rent-seekers compete for a prize (or a rent) of size V. If player $i \in J = \{1, 2, \ldots, N\}$ makes an expenditure of x_i in order to capture the prize, his share of the prize is assumed to be

$$s_i\left(x_i | x_{-i}\right) \equiv \frac{x_i^r}{\sum_{j \in J} x_j^r}, \tag{1}$$

where r can be interpreted as a parameter that summarizes the behavior of the marginal cost of influencing the share of the prize in terms of rent-seeking expenditure, and x_{-i} denotes a strategy profile with respect to all rent-seekers except for i. In Tullock's (1980) original model, the formulation in (1) is interpreted as the probability of winning the contest in which a winner obtains the entire prize. Unlike this, to avoid any unnecessary complexity in the Markov process that we will explore in subsequent sections, we assume that the rent-seeking contest is deterministic. Player i attempts to maximize his payoff:

$$\Pi_i\left(x_i|x_{-i}\right) \equiv s_i\left(x_i|x_{-i}\right)V - x_i. \tag{2}$$

When $r \leq N/\left(N-1\right)$, in a symmetric Nash equilibrium, each of these N players makes an expenditure of

$$x^{SNE} = \frac{N-1}{N^2}rV. \tag{3}$$

Therefore, the total rent-seeking expenditure in the symmetric Nash equilibrium amounts to

$$X^{SNE} \equiv Nx^{SNE} = \frac{N-1}{N}rV, \tag{4}$$

for $r \leq N/\left(N-1\right)$. However, when $r > N/\left(N-1\right)$ and finite, x^{SNE} in (3) is no longer a Nash equilibrium expenditure. It is easily verifiable that the payoff to each player in (2), after the substitution of (3), will be negative. Since the strategy x^{SNE} is dominated by a zero bid, the symmetric solution to the N players' first-order conditions does not yield a global maximum if $r > N/\left(N-1\right)$. Thus, the rate of rent dissipation X^{SNE}/V cannot be greater than one. That is, overdissipation never occurs in the symmetric Nash equilibrium of the static Tullock contest.

3 The dynamic Tullock contest

Hehenkamp et al. (2004) showed that a finite population ESS exists in the Tullock contest and the total amount of ESS expenditures results in overdissipation of the rent if $1 < r \leq N/\left(N-1\right)$. Evolutionary stability, however, does not specify a dynamic selection process that forces the attainment of a state consisting of all the ESSs, as mentioned in Section 1. Hehenkamp et al. (2004) state that the imitative behavior among rent-seekers is obviously one source of the dynamics behind ESS. In this section, we explicitly model a situation wherein imitative behavior among rent-seekers prevails in evolutionary dynamics.

3.1 Imitative learning dynamics

For technical reasons, we assume that the strategy space is a finite grid, i.e., $\Gamma = \{0, \Delta, 2\Delta, ..., z\Delta\}$ where $\Delta \in \mathbb{R}_{++}$ and $z \in \mathbb{N}$; Δ can be arbitrarily small. The only restriction imposed on Γ is that $rV/N \in \Gamma$. Rent-seekers can only observe the rent-seeking expenditures $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ and the payoffs from the contest $\mathbf{\Pi} = (\Pi_1, \Pi_2, \ldots, \Pi_N)$.

Dynamics proceed in discrete time, indexed by $t = 0, 1, 2, \ldots$. The rent-seeking expenditure of player i at t is denoted by $x_i(t)$. For brevity, we occasionally denote the payoff at period t $\Pi_i(x_i(t)\,|x_{-i}(t))$ by $\pi^i(t)$. At each t, there are two stages. In the first stage, it is determined whether or not player i revises his expenditure $x_i(t-1)$ with a common and independent probability $\delta > 0$. If player i decides to revise it, the decision process proceeds to the second stage in which player i chooses from the set

$$IM(t-1) = \left\{ x_j(t-1) \in \Gamma \mid \pi^j(t-1) = \max\left\{\pi^l(t-1)\right\}_{l \in J} \right\}, \tag{5}$$

according to an independent probability distribution with full support. We can notice that through the imitation rule defined in (5), the N-dimensional vector of expenditures at t, denoted by $\mathbf{x}(t)$, is determined by an N-dimensional vector of expenditures at $t-1$ $\mathbf{x}(t-1)$. In other words, we have a discrete-time Markov process with finite state space Γ^N.[1] Let a monomorphic state in which all rent-seekers choose the identical expenditure x be denoted by $mon(x) = (x, x, \ldots, x)$. Thus, we obtain the following results:

Lemma 1 *Any monomorphic state $mon(x)$ is in a limit set of the imitative dynamics, and any limit set is in $\{mon(x)\}_{x\in\Gamma}$.*

Proof. From the imitation rule, it is obvious that for any $x \in \Gamma$, the state $mon(x)$ leads to a limit set of the imitative dynamics. Suppose that, contrary to our latter claim, a state in a limit set is a nonmonomorphic state. Since each rent-seeker chooses any $x \in IM(t-1)$ with a positive probability, there is always a positive probability that the

[1] Our setting is in accordance with the Vega-Redondo(1997) model of learning dynamics of Cournot oligopolistic competition with a discrete-time Markov process. Recently, Alós-Ferrer, Ania and Schenk-Hoppé (2000) investigated Bertrand competition in imitative dynamics by adopting the model of Vega-Redondo (1997).

process will transit from the nonmonomorphic state to a different state. This contradicts our assumption that the nonmonomorphic state is in a limit set. ■

Therefore, there are $z+1$ singleton limit sets because the number of monomorphic states is equal to the number of available expenditures in the finite strategy space Γ.

3.2 The perturbed Markov process

We now introduce an experimentation that will occur with some independent probability $\epsilon > 0$. Once the experimentation occurs, $x_i(t) \in \Gamma$ is chosen according to some given probability distribution with full support on Γ. Thus, for each $\epsilon > 0$, the perturbed Markov process has an irreducible transition matrix such that there is a unique invariant distribution μ_ϵ that is independent of initial conditions and assigns positive probability to all states in Γ^n. The invariant distribution μ_ϵ, however, depends on the occurrence of experimentation ϵ; moreover, the invariant distribution will concentrate almost all of its probability on a few states as $\epsilon \to 0$ (see, Foster and Young [1990], Kandori, Mailath, and Rob [1993], and Young [1993]). That is, we can investigate the relative robustness among the monomorphic states as the occurrence of the experimentation vanishes. Hence, we focus on the limit invariant distribution of the perturbed Markov process as $\epsilon \to 0$, denoted by $\mu^* \equiv \lim_{\epsilon\to 0} \mu_\epsilon$. If a state is assigned some positive probability according to the limit invariant distribution μ^*, then the state is considered to be *stochastically stable*. A *stochastically stable set* consists of all the states with positive probabilities. Stochastically stable states are, intuitively, the states that are most likely to be observed over the long run when the occurrence of experimentation is rare.

According to recent evolutionary literature, we use the techniques provided by Freidlin and Wentzell (1984) to find a stochastically stable state. Any ordered pair of states is referred to as an "arrow," which is denoted by $(\mathbf{x}', \mathbf{x}'')$ for $\mathbf{x}', \mathbf{x}'' \in \Gamma^N$. For each $\mathbf{x} \in \Gamma^N$, an $\mathbf{x}$-tree is a collection of the arrows such that every $\mathbf{x}' \in \Gamma^N \setminus \{\mathbf{x}\}$ is the first element of the arrow, and for every $\mathbf{x}' \in \Gamma^N \setminus \{\mathbf{x}\}$ there is a path $\{(\mathbf{x}^0, \mathbf{x}^1), (\mathbf{x}^1, \mathbf{x}^2), \ldots, (\mathbf{x}^{s-1}, \mathbf{x}^s)\}$ where $\mathbf{x}^0 = \mathbf{x}'$ and $\mathbf{x}^s = \mathbf{x}$. The cost of the arrow $(\mathbf{x}', \mathbf{x}'')$ is the minimal number of experimentations required for the transition from $\mathbf{x}'$ to $\mathbf{x}''$ to occur with a positive probability. The cost of a $\mathbf{x}$-tree is the sum of the costs of all the arrows that belong to the $\mathbf{x}$-tree. The least cost among all $\mathbf{x}$-trees is the *stochastic potential* of $\mathbf{x}$. Young (1993) verifies that the limit invariant distribution μ^* assigns positive probability only to the states having a minimum stochastic potential. In other words, any state with a minimal cost tree among all trees of all limit states in an unperturbed Markov process is stochastically stable and a stochastically stable set consists of all the limit states with minimal cost trees.

Suppose that m $(\leq N-1)$ rent-seekers choose x while $N-m$ rent-seekers choose $\hat{x}$. Defining $\hat{x}_{-i}(m) \equiv (\underbrace{x,\ldots x}_{m}, \underbrace{\hat{x},\ldots,\hat{x}}_{N-1-m})$ and $\hat{x}_{-i}(m-1) \equiv (\underbrace{x,\ldots x}_{m-1}, \underbrace{\hat{x},\ldots,\hat{x}}_{N-m})$, the payoff for a player choosing x is

$$\Pi_i (x|\hat{x}_{-i}(m-1)) = \frac{x^r}{mx^r + (N-m)\hat{x}^r}V - x. \tag{6}$$

The payoff for a player choosing $\hat{x}$ is

$$\Pi_i (\hat{x}|\hat{x}_{-i}(m)) = \frac{\hat{x}^r}{mx^r + (N-m)\hat{x}^r}V - \hat{x}. \tag{7}$$

Subtracting (7) from (6), we have the following relative payoff:

$$\Phi(x,\hat{x};m) = \frac{x^r - \hat{x}^r}{mx^r + (N-m)\hat{x}^r}V - x + \hat{x}. \tag{8}$$

The relative payoff is useful in computing the cost of an arrow. If $\Phi(x,\hat{x};N-1) \leq 0$, then the minimal number of experimentations required for the transition from $mon(x)$ to $mon(\hat{x})$ to occur with a positive probability, i.e., the cost of the arrow, is just one. Therefore, if $\hat{x}$ is a global maximizer of $\Phi(x,\hat{x};N-1)$, then $\Phi(x,\hat{x};N-1) \leq 0$ for any $x \neq \hat{x}$. Thus, the stochastic potential of $mon(\hat{x})$ is z since for every $x \neq \hat{x}$ the cost of the arrow from $mon(x)$ to $mon(\hat{x})$ is one. Thus, $mon(\hat{x})$ has the minimum stochastic potential since z is the number of monomorphic states except for $mon(\hat{x})$.

Lemma 2 *If $r \leq N/(N-m)$ for given $m \in [1, N-1]$, then $x = rv$ is a global maximizer of $\Phi(x, rv; m)$ where $v \equiv V/N$.*

Proof. Suppose, for the moment, that player i chooses $x_i \in C$ where C is a continuum of expenditures on the interval $[0,\infty)$ in $\mathbb{R}_+$. The first-order condition for maximizing $\Phi(x,\hat{x};m)$ in (8) with respect to x is

$$\frac{\partial}{\partial x}\Phi(x,\hat{x};m) = \frac{Nx^{r-1}\hat{x}^r}{[mx^r + (N-m)\hat{x}^r]^2}rV - 1 = 0. \tag{9}$$

When $x = \hat{x} = rv$,

$$\frac{\partial}{\partial x}\Phi(rv, rv; m) = \frac{1}{Nrv}rV - 1 = 0, \tag{10}$$

thus, we obtain $x = rv$. The second-order derivative of $\Phi(x, rv; m)$ with respect to x is

$$\frac{\partial^2}{\partial x^2}\Phi(x, rv; m) = \frac{(r-1)(N-m)(rv)^r - (r+1)mx^r}{[mx^r + (N-m)(rv)^r]^3}x^{r-2}(rv)^r NrV. \tag{11}$$

Therefore, the sufficient condition for rv locally maximizing $\Phi(x, rv; m)$ is

$$\lambda \equiv \frac{(r-1)(N-m)}{(r+1)m} < 1. \tag{12}$$

It is evident from (11) that the function $\Phi(x, rv; m)$ is strictly concave (respectively, convex) if $x >$ (respectively, $<$) $\lambda^{\frac{1}{r}}rv$. For $r \leq 1$, $\Phi(x, rv; m)$ is concave at $x = 0$ and strictly concave at any $x > 0$ since $\lambda \leq 0$. Hence, $x = rv$ is a global maximizer of $\Phi(x, rv; m)$ for any $r \leq 1$.

Next, consider the case in which $r > 1$. When $r > 1$, λ is positive such that $\Phi(x, rv; m)$ is strictly convex for any $x < \lambda^{\frac{1}{r}}rv$ and strictly concave for any $x > \lambda^{\frac{1}{r}}rv$. Thus, $\lambda^{\frac{1}{r}}rv$ is the only inflection point of $\Phi(x, rv; m)$. This implies that either 0 or rv can globally maximize $\Phi(x, rv; m)$ if the inequality of (12) is satisfied, i.e., $\lambda < 1$. When $m \geq N/2$, (12) holds regardless of r. Rearranging (12) yields $r < N/(N-2m)$ if $m < N/2$. That is, (12) can be satisfied for $r < N/(N-2m)$ when $m < N/2$. Substituting 0 and rv for x and $\hat{x}$ in (8), respectively, we obtain

$$\Phi(0, rv; m) = \frac{r(N-m) - N}{N(N-m)}V. \tag{13}$$

For any $r \leq N/(N-m)$, we have $\Phi(0, rv; m) \leq 0$. Since $\lambda < 1$ for any $r < N/(N-2m)$ and $\Phi(rv, rv; m) = 0$, rv is a global maximizer of $\Phi(x, rv; m)$ for any $r \leq N/(N-m)$. Figure 1 depicts a configuration of the graph of $\Phi(x, rv; m)$ in the case that $1 < r < N/(N-m)$. From the assumption that rv is contained in the finite grid Γ, rv can also be a global maximizer of $\Phi(x, rv; m)$ when a strategy space is Γ. ■

Lemma 2 states that a player choosing rv acquires the highest payoff when the number of other players choosing $x \neq rv$, denoted by m, is such that $r \leq N/(N-m)$.

Proposition 1 *For $r \leq N$, there exists a unique stochastically stable state in the imita-*

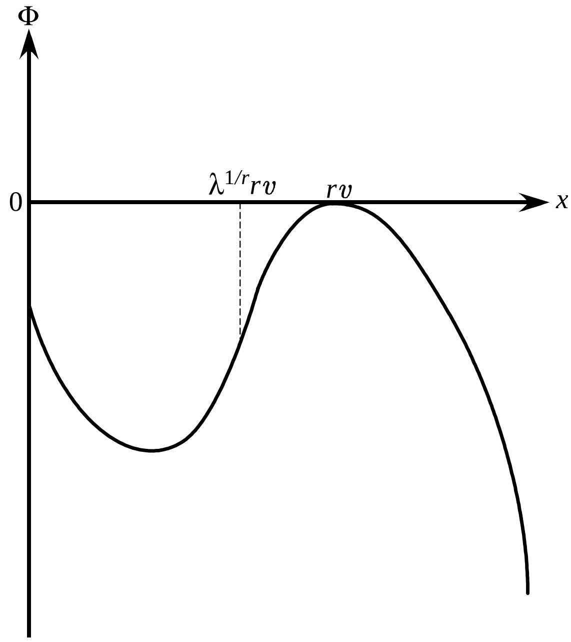

Figure 1: Configuration of the graph of the function $\Phi\left(x, rv; m\right)$

tive learning dynamics of the Tullock contest, which is given by $mon\left(rv\right)$.

Proof. Lemma 1 states that no nonmonomorphic state can be included in the stochastically stable set. When $m = N - 1$, from Lemma 2, rv is a global maximizer of $\Phi\left(x, rv; N-1\right)$ if $r \leq N$. Thus, we have

$$\Phi\left(x, rv; N-1\right) \leq 0 \text{ for any } x \in \Gamma \setminus \left\{rv\right\}. \tag{14}$$

The cost of $mon\left(rv\right)$-tree is z since the inequality (14) implies that for every $x \in \Gamma \setminus \left\{rv\right\}$, the cost of the arrow from $mon\left(x\right)$ to $mon\left(rv\right)$ is just one. Therefore, the stochastic potential of $mon\left(rv\right)$ is equal to z. Clearly, z is the minimum stochastic potential. Hence, $mon\left(rv\right)$ is contained in the stochastically stable set.

For uniqueness, we must verify that no tree rooted in $mon\left(x\right)$ except for $mon\left(rv\right)$ has the minimum stochastic potential z. Consider first $mon\left(x\right)$ for any $x \geq V$. In this case, the cost of the arrow from $mon\left(0\right)$ to $mon\left(x\right)$ is more than one since $\Phi\left(0, x; N-1\right) \geq 0$. Hence, $mon\left(x\right)$ for any $x \geq V$ cannot be a stochastically stable state. Next, consider $mon\left(x\right)$ for any $x \in \left(0, V\right) \setminus \left\{rv\right\}$. We can easily observe that $\Phi\left(0, x'; N-1\right) < 0$ and $\frac{\partial}{\partial x}\Phi\left(0, x'; N-1\right) = -1$, where $x' \in \left(0, V\right)$. In addition, $\Phi\left(x, x'; N-1\right)$ is strictly convex for $x < \lambda^{\frac{1}{r}} x'$ and strictly concave for $x > \lambda^{\frac{1}{r}} x'$ if $r \leq N$. Further, $\Phi(x', x'; N-1) = 0$ and we obtain the following from (10):

$$\frac{\partial}{\partial x} \Phi(x', x'; N-1) = \frac{1}{Nx'} rV - 1 \gtreqless 0 \text{ if } x' \lesseqgtr rv. \tag{15}$$

Accordingly, we obtain $\Phi(x' + \Delta, x'; N-1) > 0$ for any $x' < rv$ and $\Phi(x' - \Delta, x'; N-1) > 0$ for any $x' > rv$ by regarding Δ as a sufficiently small value. That is, the costs of the arrows from $mon(x' + \Delta)$ to $mon(x')$ for any $x' < rv$ and from $mon(x' - \Delta)$ to $mon(x')$ for any $x' > rv$ are both more than one. Thus, for any positive $x' \neq rv$, the stochastic potential of a tree rooted in $mon(x')$ is more than z.

Finally, the tree rooted in $mon(0)$ cannot have the minimum stochastic potential since there exists a positive x such that $\Phi(x, 0; N-1) \geq 0$. ■

For any positive $r \leq N$, the only stochastically stable state $mon(rv)$ leads to the aggregate rent-seeking expenditure of $Nrv = rV$. Thus, the rate of rent-dissipation in the stochastically stable state is equal to r. Therefore, when r is greater than one but not greater than N, overdissipation occurs frequently in the long run where the stochastically stable state can be overwhelmingly attained.

Hehenkamp et al. (2004) show that there exists a unique ESS for any positive $r \leq N/(N-1)$. The ESS derived is identical to the individual strategy rv constituting the stochastically stable state derived above. The set of r ensuring the existence of the ESS is, however, included in the set of r for our stochastically stable state, except for the case in which $N = 2$. In other words, when $N/(N-1) < r \leq N$, rv is the strategy constituting the SSS but not an ESS. This difference increases in contrast as the number of rent-seekers increases. That is, the larger the N, the larger is the supremum of the set of r for the SSS and the smaller is the supremum for the ESS. Moreover, as $N \to \infty$, the supremum for the ESS converges to 1; thus, the possibility of overdissipation in the ESS vanishes. However, in our imitative dynamics, the infinitely large number of rent-seekers may result in a higher rate of overdissipation since it allows the existence of the SSS rv for any large r.

The ESS in Hehenkamp et al. (2004), however, is a finite population ESS that is defined in Schaffer (1988), and therefore, N is assumed to be finite such that there is a range of r that induces overdissipation. It is necessary to satisfy the following condition in order that rv is a finite population ESS in the Tullock contest.

$$\Phi(x, rv; 1) < 0 \text{ for any } x \neq rv. \tag{16}$$

That is, if a single rent-seeker chooses any strategy $x \neq rv$ but other $N-1$ rent-seekers choose rv, then the payoff for the rent-seeker choosing $x \neq rv$ must be lower than the payoffs for the other rent-seekers. Since the only candidates for a global maximum of $\Phi(x, rv; 1)$ are 0 and rv, the ESS requires that a single rent-seeker choosing a zero bid obtains a strictly lower payoff than the other $N-1$ rent-seekers who choose rv. In other words, each of $N-1$ rent-seekers choosing rv must obtain a positive payoff, i.e., $Nv/(N-1) - rv > 0$, which yields $r < N/(N-1)$. In order that rv constitutes a stochastically stable state given zero bids for $N-1$ rent-seekers, a single rent-seeker choosing rv only has to obtain a nonnegative payoff, while the other $N-1$ rent-seekers retain zero payoffs, as denoted in (13). Therefore, it should be required that $Nv - rv \geq 0$ or equivalently $r \leq N$.

Alós-Ferrer and Ania (2005) reveal that in a rent-seeking contest, a profile consisting of N identical ESSs, which are globally stable, coincides with the unique SSS of the imitative dynamics with experimentation if the globally stable ESS exists. However, they do not investigate whether or not the converse holds. Global stability of an ESS requires that a player choosing the ESS has a higher payoff than each of m players choosing any other strategy for every $m \in [1, N-1]$. We can easily see from Lemma 2 that for any $m \in [1, N-1]$ and $x \neq rv$, $\Phi(x, rv; m) < 0$ if $r < N/(N-1)$, which implies that rv is a globally stable ESS for any $r < N/(N-1)$. That is, the necessary condition for rv to be a globally stable ESS is $r < N/(N-1)$. However, $r < N/(N-1)$ is not necessary but sufficient for $mon(rv)$ to be a unique SSS. The result we derived provides one of the examples in which the strategy constituting a unique SSS is not a globally stable ESS since for any $r \in (N/(N-1), N]$, $mon(rv)$ is the unique SSS but rv is not an ESS.

4 The modified imitative learning rule

In the imitation rule described in Section 3.1, even though the *absolute* payoff for the most successful rent-seeker is negative, the other rent-seekers will imitate the choice of the most successful rent-seeker with a positive probability in the next period. This imitation rule, however, does not seem to be plausible. Even rent-seekers without full information and computation ability would know *a priori* that their absolute payoff can be zero regardless of the other players' strategies if they choose zero expenditure. Therefore, the rent-seekers may infer that the most successful *strategy* should not be one of observable positive expenditures but zero expenditure. Taking this into account, it is worth investigating the following modified imitation rule: Player i chooses from the set

$$MIM\,(t-1) = \left\{x_j\,(t-1) \in \Gamma \mid \pi^j\,(t-1) = \max\left\{\pi^l\,(t-1)\right\}_{l\in J} \ \&\ \pi^j\,(t-1) \geq 0\right\}, \tag{17}$$

according to an independent probability distribution with full support if $MIM\,(t-1) \neq \varnothing$; otherwise, $x_i\,(t) = 0$. In other words, the rent-seekers will decide to revise and imitate an expenditure among observations only when they find one that results in a positive payoff. If not, they would abstain from rent-seeking activities. We can refer to $\pi^j\,(t-1) \geq 0$ appearing in (17) as "individual rationality" or "participation" constraint because the reservation payoff for each rent-seeker is equal to zero.[2] Additionally, we assume that v is included in the finite grid Γ.

Proposition 2 *Under the modified imitation rule, a stochastically stable set in the imitative learning dynamics of the Tullock contest is given by:*
(i) $\{mon\,(rv)\}$ *for any positive* $r \leq 1$
(ii) $\{mon\,(0)\,, mon\,(v)\}$ *for any finite* $r > 1$.

Proof. See Appendix. ■

A monomorphic state $mon\,(x)$ for any $x > v$ is excluded from the limit sets of the unperturbed Markov process owing to the modified imitation rule since any strategy $x > v$ conflicts with individual rationality for each rent-seeker. A violation of individual rationality entails overdissipation in the contest. Therefore, under the modified imitation rule, which is compatible with individual rationality, overdissipation cannot prevail in the long run. In the stochastically stable state $mon\,(v)$ for $r \geq 1$, each rent-seeker expends v to win a share of the rent. Consequently, the sum of the expenditures in the per-period contest amounts to V; thus, the rent V is fully dissipated for any finite $r \geq 1$ in the stochastically stable state. In the other stochastically stable state $mon\,(0)$, however, the rent is not dissipated at all since all rent-seekers are inactive.

[2] We employ the term of individual rationality in the meaning that potential players participate in a game only when their payoffs from the game will be more than or equal to their reservation payoff.

Remark 1 *Notice that* $\mathrm{mon}\,(v)$ *is not stochastically unstable even though a transition from* $\mathrm{mon}\,(v)$ *to some state occurs with only one experimentation. Consider, for instance, the case in which a player chooses* $v + \Delta$ *by way of experimentation in the state of* $\mathrm{mon}\,(v)$*. In this case, we have* $\Phi\,(v + \Delta, v; 1) > 0$ *by taking* Δ *as a sufficiently small value. This implies that the transition from* $\mathrm{mon}\,(v)$ *to* $\mathrm{mon}\,(v + \Delta)$ *can be realized with one experimentation. However,* $\mathrm{mon}\,(v + \Delta)$ *is transient and thus can move to* $\mathrm{mon}\,(0)$ *without experimentation owing to the modified imitation rule. That is, the transition from* $\mathrm{mon}\,(v)$ *to* $\mathrm{mon}\,(0)$ *can occur with one experimentation. Further, the transition from* $\mathrm{mon}\,(0)$ *to* $\mathrm{mon}\,(v)$ *can occur with one experimentation. In short, we must not judge a state as being unstable because the state can move to each of some other states with one experimentation. In this respect, stochastic stability is different from the concept of a finite population ESS.*

Remark 2 *Since the concept of a finite population ESS does not express a dynamic process, our criticism regarding individual rationality could not directly apply to the result in the finite population ESS of the Tullock contest. Nevertheless, we might infer that the ESS could be obtained as a consequence of the simple imitation rule such as (5) because the ESS coincides with the strategy constituting the SSS with the simple imitation rule for any* $r < N/\,(N-1)$*. Therefore, we can state that the concept of the finite population ESS ignores individual rationality in the Tullock contest. Hehenkamp et al. (2004) demonstrate that the finite population ESS coincides with a symmetric Nash equilibrium strategy when each player maximizes his relative payoff assuming strategies of all the other players as given. Because the relative payoff becomes zero at the ESS, we may interpret that individual rationality is satisfied with respect to the relative payoff and therefore not violated in the evolutionary equilibrium. However, the objective of maximizing a relative payoff is a consequence of an economic natural selection in which the dynamic forces would be driven by the simple imitation rule. Our concern regarding individual rationality is not in the consequence but in the source of the economic natural selection. If the objective of relative payoff maximization is a consequence of imitative learning dynamics conflicting with individual rationality, we suspect that the consequence may not be valid because the premises of the model are implausible.*

4.1 State-dependent experimentations

In the modified imitation rule, for any finite $r > 1$ we have two stochastically stable states: $mon\,(0)$ and $mon\,(v)$. The states that will be superior over the other in the long run can be determined if the experimentation process can be modified in a more economically justifiable manner such that experimentation rates are state-dependent.[3]

When experimentation rates are state-dependent, the cost of transition from $\mathbf{x}'$ to $\mathbf{x}''$ need not be the minimal number of experimentations required for the transition. Let $P^{\epsilon}(\mathbf{x}', \mathbf{x}'')$ denote the transition probability from $\mathbf{x}'$ to $\mathbf{x}''$ in a perturbed Markov process. The cost of the transition from $\mathbf{x}'$ to $\mathbf{x}''$ is, more formally, represented as $c(\mathbf{x}', \mathbf{x}'') \geq 0$ such that $0 < \lim_{\epsilon \to 0} P^{\epsilon}(\mathbf{x}', \mathbf{x}'') / \epsilon^{c(\mathbf{x}', \mathbf{x}'')} < \infty$. If the number of experimentations necessary for the transition from $\mathbf{x}'$ to $\mathbf{x}''$ is at least $k \in \mathbb{N} \cup \{0\}$ and the experimentation rate in the state $\mathbf{x}'$ is ϵ^{α} where α is a positive parameter, then $P^{\epsilon}(\mathbf{x}', \mathbf{x}'')$ is on the same order of $\epsilon^{k\alpha}$. Thus, we obtain $c(\mathbf{x}', \mathbf{x}'') = k\alpha$. We can easily observe that the cost of the transition from $\mathbf{x}'$ to $\mathbf{x}''$ equals to the minimal number of experimentations required for the transition only when $\alpha = 1$.

We assume that the experimentation rate in $mon\,(0)$ is ϵ^{α} while each rate in all the other states is still ϵ. In other words, the probability that each rent-seeker will experiment in the state where all rent-seekers are inactive can be different from that in any monomorphic state where all of them engage in rent-seeking activities.

Proposition 3 *Assume that the experimentation rate in* $mon\,(0)$ *is* ϵ^{α} *while the one in* $mon\,(x)$ *for any* $x \in (0, v]$ *is* ϵ. *Under the modified imitation rule, for any finite* $r > 1$, *there exists a unique stochastically stable state in the imitative learning dynamics of the Tullock contest, which is given by* $mon\,(0)$ *if* $\alpha > 1$ *and* $mon\,(v)$ *if* $0 < \alpha < 1$.

Proof. The cost of the transition from $mon\,(0)$ to $mon\,(v)$ is α since the minimal number of experimentations required for the transition is one and the experimentation rate in $mon\,(0)$ is ϵ^{α}. Hence, the stochastic potential of $mon\,(v)$-tree is $\zeta - (1 - \alpha)$. We have shown in the proof of Proposition 2 that any transition from $mon\,(v)$ to $mon\,(x)$ for every $x \in (0, v)$ involves more than one cost. Therefore, the stochastic potential of $mon\,(x)$-tree

[3] Bergin and Lipman (1996) show that any prediction can be attained depending on the restrictions on how experimentation rates vary across states that are contained in a limit set of an unperturbed dynamic. This result leads to two implications. One is that the refinement of multiple long-run equilibria by introducing experimentations is meaningless. The other is that we should derive an economically justifiable restriction on the experimentation process.

for every $x \in (0, v)$ is more than $\zeta - (1 - \alpha)$. That is, $mon\,(x)$ for any $x \in (0, v)$ is not a stochastically stable state regardless of α. Since the experimentation rate in $mon\,(x)$ for any $x \in (0, v]$ is assumed to be ϵ, the stochastic potential of $mon\,(0)$-tree is ζ. Hence, compared with the stochastic potential of $mon\,(v)$-tree, the one of $mon\,(0)$-tree is smaller if $\alpha > 1$ and greater if $0 < \alpha < 1$. ■

When the highest payoff observed in the contest is negative, in the modified imitative learning rule, each rent-seeker realizes that there is nothing to learn by imitation and thus chooses zero expenditure. Therefore, in the state where all rent-seekers are inactive, i.e., $mon\,(0)$, we should regard that each rent-seeker has nothing to learn by imitation because he knows *a priori* that zero expenditure yields zero payoff for him regardless of the other rent-seekers' choices. Since the rent-seekers cannot, in fact, imitate the others, the experimentation rate in $mon\,(0)$ would be higher compared with any monomorphic state where all rent-seekers engage in rent-seeking activities. In addition, a rent-seeker would tend to make a positive expenditure arbitrarily if nobody tries to win the prize that hangs in before him. Therefore, it would be reasonable for us to assume that $\alpha < 1$, whereas it is difficult to justify the assumption that $mon\,(v)$ involves a higher experimentation rate than $mon\,(0)$. This is because the rent-seekers are able to learn by imitation in $mon\,(v)$ if they so desire.

If the rent-seekers have to pay a fixed amount of entry fee when they revise their zero expenditure, the experimentation rate in $mon\,(0)$ may decrease as compared with the case in which no entry fee is required. The reason for this is that the presence of the fixed entry fee would discourage the rent-seekers from changing the status quo in $mon\,(0)$. When $r > 1$ and the entry fee is sufficiently large such that $\alpha > 1$, socially wasteful rent-seeking expenditures cannot be observed at all in the states that prevail in the long run.

5 Conclusions

The evolutionary dynamics behind a finite population ESS in the Tullock contest, as mentioned in Hehenkamp et al. (2004), are mainly driven by imitative behavior among rent-seekers. We have modeled such imitative behavior explicitly in the dynamic Tullock contest and have demonstrated that the unique stochastically stable state is consistent with the finite population ESS derived in Hehenkamp et al. (2004). Moreover, the stochastically stable state can involve a higher rate of overdissipation than the finite population ESS.

One source of overdissipation is the lack of individual rationality. In other words, without individual rationality rent-seekers engage in the contest even though they incur a negative payoff, and thus, the sum of the individual expenditures is more than the value of the rent. However, the notion of the ESS cannot exclude any imitative behavior that results in a negative individual payoff, although it is difficult to find such irrational behavior in any real life contest. The finite population ESS derived by Hehenkamp et al. (2004) does not satisfy individual rationality for any r in the interval of $(1, N/(N-1)]$ such that it yields overdissipation. Therefore, in order that the ESS is compatible with individual rationality, our analyses must be restricted to the case in which $r \leq 1$, whereas by reconciling individual rationality with imitation of the most successful behavior, we have shown that there exists a stochastically stable state for any finite r.

Corcoran (1984) and Corcoran and Karels (1985) have studied a long-run equilibrium of the rent-seeking contest endogenizing the entry decisions of rent-seekers. They showed that the number of rent-seekers who have decided to participate in the contest increases or decreases until the payoffs for the active rent-seekers fall to zero to the extent that $1 < r \leq N/(N-1)$. Therefore, in the long-run equilibrium of their free entry model, for any $r \in (1, N/(N-1)]$, individual rationality or participation constraints are satisfied, and thus, the rent is fully dissipated. Itaya and Sano (2003) have examined the long-run behavior of rent-seekers, each of whom chooses a mixed strategy for deciding whether to stay in or exit from the Tullock contest in the multi-period setting. They verified that for some $r > N/(N-1)$, overdissipation can arise in each period[4] while the sum of the present discounted values of the per-period expected payoffs is equal to zero, i.e., the sum of the present discounted values of the rents will be fully dissipated over time *ex ante*. In this paper, we have shown that for any finite $r \geq 1$, the rent is fully dissipated in one of two stochastically stable states under the modified imitative learning rule. This implies that in the long run, the occurrence of overdissipation is rare and the "full-dissipation hypothesis" may still be valid even though r is greater than 1.

Since rent-seeking competition may entail some illegal activities such as bribes, each rent-seeker would not be able to observe expenditures of the other rent-seekers. In such

[4] Itaya et al. (2003) introduced a minimum expenditure that is required to achieve the positive probability of winning the contest. Owing to the minimum expenditure, exiting decisions of rent-seekers can be compatible with individual rationality.

a case, it is not appropriate to assume the imitative learning behavior of rent-seekers, and therefore, we should assume that the rent-seekers are learning only from their own experiences, which is known as *introspective learning*. With an introspective learning rule, we may obtain a long-run equilibrium result greatly different from the results that we have derived in this paper.[5] This issue will be approached in future research.

Appendix

Proof of Proposition 2. Clearly, no nonmonomorphic state belongs to a limit set of the unperturbed Markov process in the modified imitation rule. Moreover, any monomorphic state $mon\,(x)$ such that $x > v$ yields a negative individual payoff, i.e., $\pi_i\,(t-1) = v - x < 0$. Since $MIM\,(t-1) = \varnothing$ when $\pi_i\,(t-1) < 0$, player i will choose $x_i = 0$ with a positive probability in period t. Therefore, for every $x > v$, $mon\,(x)$ is not in a limit set of the unperturbed Markov process. Thus, the only candidates for a stochastically stable state are monomorphic states where all players choose the identical expenditure that belongs to the discrete interval $[0, v] \subset \Gamma$. Without loss of generality, the number of strategies in $[0, v]$ is assumed to be $\zeta + 1$ such that $1 \leq \zeta < z$.

Consider first the case in which $0 < r \leq 1$. In this case, $mon\,(rv)$ results in a nonnegative payoff. In addition, from Lemma 2, rv is a global maximizer of $\Phi\,(x, rv; m)$ for any $r \leq 1$. Hence, $mon\,(rv)$ is a unique stochastically stable state when $r \leq 1$.

Next, consider the case in which $1 < r < \infty$. From (8) and (9), we have $\Phi\,(0, v; m) = -mv/\,(N-m) < 0$ and $\frac{\partial}{\partial x}\Phi\,(0, v; m) = -1$, respectively. Moreover, $\Phi\,(x, v; m)$ is strictly convex for $x < \lambda^{\frac{1}{r}} v$ and strictly concave for $x > \lambda^{\frac{1}{r}} v$. Therefore, $\Phi\,(x, v; m) = 0$ has at most two real roots. Since $r > 1$, from (9) we obtain $\frac{\partial}{\partial x}\Phi\,(v, v; m) = r - 1 > 0$ regardless of m. This implies that $x = v$ is the smaller one of the two real roots as represented in Figure 2. Hence, we obtain

$$\Phi\,(x, v; N-1) < 0 \text{ for any } x \in [0, v)\,. \tag{A1}$$

Thus, any transition from $mon\,(x)$ to $mon\,(v)$ for any $x \in [0, v)$ requires only one cost. Accordingly, the cost of a tree rooted in $mon\,(v)$ is ζ such that $mon\,(v)$ is contained in a

[5] Bergin and Bernhardt (2004) have investigated both cases in their settings. They claim that introspective learning is the key condition to ensure that a dynamic system converges to Nash equilibrium states.

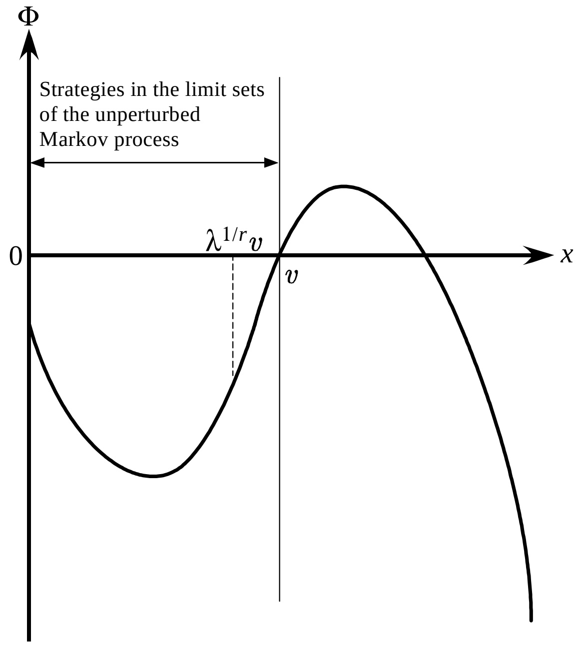

Figure 2: Configuration of the graph of the function $\Phi(x, v; m)$ for $r > 1$

stochastically stable set of the perturbed Markov process. Further, we have

$$\Phi(x, v; 1) < 0 \text{ for any } x \in (0, v). \tag{A2}$$

This implies that any transition from $mon(v)$ to $mon(x)$ for every $x \in (0, v)$ involves more than one cost. Hence, $mon(x)$ for any $x \in (0, v)$ never belongs to the stochastically stable set.

Finally, we demonstrate that $mon(0)$ is another stochastically stable state. Consider a monomorphic state $mon(x')$ for any $x' \in (0, v]$. From (14), we have $\Phi(x', rv; N-1) \leq 0$ for any $r \in (1, N]$. That is, the minimal number of experimentations necessary for the transition from $mon(x')$ to $mon(rv)$ is one when $1 < r \leq N$. Since rv is contained in the basin of attraction of $mon(0)$, the cost of the arrow from $mon(x')$ to $mon(0)$ is one for any $r \in (1, N]$. Next, suppose that $N < r < \infty$. Then, we obtain from (9), for any $m \in [1, N-1]$,

$$\frac{\partial}{\partial x}\Phi(v+\Delta, v+\Delta; m) = \frac{(r-1)v - \Delta}{v+\Delta}. \tag{A3}$$

Since v is assumed to be contained in Γ and $r > N$, we have $(r-1)v > (N-1)v > v \geq \Delta$; thus, (A3) is positive for any finite $r > N$. In addition, the fact that $(N-1)v$

$> \Delta$ yields $\Phi(0, v+\Delta; N-1) = -(N-1)v + \Delta < 0$. Therefore, for any $x' \in (0, v]$, $\Phi(x', v+\Delta; N-1) < 0$ since $\Phi(v+\Delta, v+\Delta; N-1) = 0$ and $\Phi(0, v+\Delta; N-1) < 0$ together with the only one inflection point of $\Phi(x, v+\Delta; N-1)$. This implies that for any $x' \in (0, v]$ the cost of the arrow from $mon(x')$ to $mon(v+\Delta)$ is just one. Since $mon(v+\Delta)$ is in the basin of attraction of $mon(0)$, $mon(0)$ also has the minimum stochastic potential ζ for any finite $r > N$. ■

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