

## Supplement to the papers on “the polyserial correlation coefficients” and “discrepancy functions for general covariance structures”

Haruhiko Ogasawara

Parts A and B of this note are to supplement Ogasawara (2011, 2010), respectively.

### Part A

#### 0. Derivation of the inverse expansion

Let  $\mathbf{L} = \partial^2 \bar{l} / \partial \boldsymbol{\theta}_0 \partial \boldsymbol{\theta}_0'$ . Then, from (3.1) the inverse expansion of  $\hat{\boldsymbol{\theta}}$  is

$$\begin{aligned} \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0 &= -\mathbf{L}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} - \frac{1}{2} \mathbf{L}^{-1} \frac{\partial^3 \bar{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0')^{<2>}} (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)^{<2>} - \frac{1}{6} \mathbf{L}^{-1} \frac{\partial^4 \bar{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0')^{<3>}} (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)^{<3>} \\ &\quad + O_p(N^{-2}) \\ &= -\mathbf{L}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} - \frac{1}{2} \mathbf{L}^{-1} \frac{\partial^3 \bar{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0')^{<2>}} \left( \mathbf{L}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)^{<2>} \\ &\quad - \frac{1}{2} \mathbf{L}^{-1} \frac{\partial^3 \bar{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0')^{<2>}} \left[ \left( \mathbf{L}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right) \otimes \left\{ \mathbf{L}^{-1} \frac{\partial^3 \bar{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0')^{<2>}} \left( \mathbf{L}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)^{<2>} \right\} \right] \\ &\quad + \frac{1}{6} \frac{\partial^4 \bar{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0')^{<3>}} \left( \mathbf{L}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)^{<3>} + O_p(N^{-2}). \end{aligned}$$

Let  $\mathbf{E}(\mathbf{L}) = \boldsymbol{\Lambda} = -\mathbf{I}$ , where  $\mathbf{I}$  is the information matrix per observation, and let

$\mathbf{L} = \boldsymbol{\Lambda} + \mathbf{M}$ , where  $\mathbf{M} = O_p(N^{-1/2})$ . Then, we obtain

$$\mathbf{L}^{-1} = \boldsymbol{\Lambda}^{-1} - \boldsymbol{\Lambda}^{-1} \mathbf{M} \boldsymbol{\Lambda}^{-1} + \boldsymbol{\Lambda}^{-1} \mathbf{M} \boldsymbol{\Lambda}^{-1} \mathbf{M} \boldsymbol{\Lambda}^{-1} + O_p(N^{-3/2}).$$

The above results give

$$\begin{aligned}
\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0 &= \left[ -\boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right]_{O_p(N^{-1/2})} \\
&+ \left[ \boldsymbol{\Lambda}^{-1} \mathbf{M} \boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} - \frac{1}{2} \boldsymbol{\Lambda}^{-1} \mathbf{E} \left( \frac{\partial^3 \bar{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0')^{<2>}} \right) \left( \boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)^{<2>} \right]_{O_p(N^{-1})} \\
&+ \left[ -\boldsymbol{\Lambda}^{-1} \mathbf{M} \boldsymbol{\Lambda}^{-1} \mathbf{M} \boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} + \frac{1}{2} \boldsymbol{\Lambda}^{-1} \mathbf{M} \boldsymbol{\Lambda}^{-1} \mathbf{E} \left( \frac{\partial^3 \bar{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0')^{<2>}} \right) \left( \boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)^{<2>} \right. \\
&\quad \left. + \boldsymbol{\Lambda}^{-1} \mathbf{E} \left( \frac{\partial^3 \bar{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0')^{<2>}} \right) \left\{ \left( \boldsymbol{\Lambda}^{-1} \mathbf{M} \boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right) \otimes \left( \boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right) \right\} \right. \\
&\quad \left. - \frac{1}{2} \boldsymbol{\Lambda}^{-1} \left\{ \frac{\partial^3 \bar{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0')^{<2>}} - \mathbf{E} \left( \frac{\partial^3 \bar{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0')^{<2>}} \right) \right\} \left( \boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)^{<2>} \right. \\
&\quad \left. - \frac{1}{2} \boldsymbol{\Lambda}^{-1} \mathbf{E} \left( \frac{\partial^3 \bar{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0')^{<2>}} \right) \left[ \left( \boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right) \otimes \left\{ \boldsymbol{\Lambda}^{-1} \mathbf{E} \left( \frac{\partial^3 \bar{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0')^{<2>}} \right) \left( \boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)^{<2>} \right\} \right] \right. \\
&\quad \left. + \frac{1}{6} \boldsymbol{\Lambda}^{-1} \mathbf{E} \left( \frac{\partial^4 \bar{l}}{\partial \boldsymbol{\theta}_0 (\partial \boldsymbol{\theta}_0')^{<3>}} \right) \left( \boldsymbol{\Lambda}^{-1} \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0} \right)^{<3>} \right]_{O_p(N^{-3/2})} + O_p(N^{-2}) \\
&\equiv \sum_{i=1}^3 \boldsymbol{\Lambda}^{(i)} \mathbf{I}_0^{(i)} + O_p(N^{-2}),
\end{aligned}$$

where  $[\cdot]_{O_p(\cdot)}$  indicates that the sum of the terms in brackets is of order  $O_p(\cdot)$  for clarity.

## 1. Expectations of the log likelihood derivatives

Let  $\boldsymbol{\Lambda} = \mathbf{B} \text{diag}(\boldsymbol{\Lambda}_x, \boldsymbol{\Lambda}_z)$  and  $\mathbf{I} = \mathbf{B} \text{diag}(\mathbf{I}_x, \mathbf{I}_z)$ , where  $\boldsymbol{\Lambda} = -\mathbf{I}$ ,  $\mathbf{B} \text{diag}(\cdot)$  denotes the block diagonal matrix with the diagonal blocks being the matrices in parentheses, and  $\mathbf{I}$  is the information matrix per observation.

### 1.1 Information matrix

#### (1) $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$

$$\frac{\partial l_x^*}{\partial \boldsymbol{\mu}} = \sum_{i=1}^N \boldsymbol{\Sigma}^{-1}(\mathbf{x}_i - \boldsymbol{\mu}),$$

$$\frac{\partial l_x^*}{\partial \sigma_{ab}} = \frac{2 - \delta_{ab}}{2} \sum_{i=1}^N [-\sigma^{ab} + \{\boldsymbol{\Sigma}^{-1}(\mathbf{x}_i - \boldsymbol{\mu})\}_a \{\boldsymbol{\Sigma}^{-1}(\mathbf{x}_i - \boldsymbol{\mu})\}_b],$$

$$\frac{\partial^2 l_x^*}{\partial \boldsymbol{\mu} \partial \boldsymbol{\mu}'} = -N \boldsymbol{\Sigma}^{-1},$$

$$\frac{\partial^2 l_x^*}{\partial \sigma_{ab} \partial \boldsymbol{\mu}} = -\frac{2 - \delta_{ab}}{2} \sum_{i=1}^N [(\boldsymbol{\Sigma}^{-1})_{.a} \{\boldsymbol{\Sigma}^{-1}(\mathbf{x}_i - \boldsymbol{\mu})\}_b + (\boldsymbol{\Sigma}^{-1})_{.b} \{\boldsymbol{\Sigma}^{-1}(\mathbf{x}_i - \boldsymbol{\mu})\}_a],$$

$$\begin{aligned} \frac{\partial^2 l_x^*}{\partial \sigma_{ab} \partial \sigma_{cd}} = & \frac{1}{4} (2 - \delta_{ab})(2 - \delta_{cd}) \sum_{i=1}^N \left[ \sigma^{ac} \sigma^{db} + \sigma^{ad} \sigma^{cb} \right. \\ & \left. - \sum_{k=1}^4 \sigma^{ac} \{\boldsymbol{\Sigma}^{-1}(\mathbf{x}_i - \boldsymbol{\mu})\}_d \{\boldsymbol{\Sigma}^{-1}(\mathbf{x}_i - \boldsymbol{\mu})\}_b \right], \end{aligned}$$

where  $\sum^k$  denotes the sum of  $k$  terms with similar patterns. Then, we obtain

$$-N(\mathbf{I}_x)_{\boldsymbol{\mu}\boldsymbol{\mu}'} = N(\boldsymbol{\Lambda}_x)_{\boldsymbol{\mu}\boldsymbol{\mu}'} = \mathbf{E} \left( \frac{\partial^2 l_x^*}{\partial \boldsymbol{\mu} \partial \boldsymbol{\mu}'} \right) = \frac{\partial^2 l_x^*}{\partial \boldsymbol{\mu} \partial \boldsymbol{\mu}'} = -N \boldsymbol{\Sigma}^{-1},$$

$$-N(\mathbf{I}_x)_{\sigma_{ab} \boldsymbol{\mu}'} = 0,$$

$$\begin{aligned} -N(\mathbf{I}_x)_{\sigma_{ab} \sigma_{cd}} = N(\boldsymbol{\Lambda}_x)_{\sigma_{ab} \sigma_{cd}} = & \mathbf{E} \left( \frac{\partial^2 l_x^*}{\partial \sigma_{ab} \partial \sigma_{cd}} \right) = -\frac{N}{4} (2 - \delta_{ab})(2 - \delta_{cd}) \\ & \times (\sigma^{ac} \sigma^{db} + \sigma^{ad} \sigma^{cb}), \end{aligned}$$

that is,

$$(\boldsymbol{\Lambda}_x)_{\boldsymbol{\mu}\boldsymbol{\mu}'} = -\boldsymbol{\Sigma}^{-1}, \quad (\boldsymbol{\Lambda}_x)_{\boldsymbol{\mu} \sigma_{ab}} = 0,$$

$$(\boldsymbol{\Lambda}_x)_{\sigma_{ab} \sigma_{cd}} = -\frac{1}{4} (2 - \delta_{ab})(2 - \delta_{cd}) (\sigma^{ac} \sigma^{db} + \sigma^{ad} \sigma^{cb}),$$

$$(r \geq a \geq b \geq 1; r \geq c \geq d \geq 1)$$

where, e.g.,  $(\cdot)_{\mathbf{u}\mathbf{v}'}$  denotes the submatrix of the matrix in parentheses corresponding to the product  $\mathbf{u}\mathbf{v}'$ , and the ranges of  $(a, b)$ ,  $(c, d)$ , and similar pairs will be used hereafter in similar situations.

(2)  $\boldsymbol{\beta}=(\beta_1,\dots,\beta_{K-1})'$  and  $\boldsymbol{\xi}=(\xi_1,\dots,\xi_r)'$

$$\frac{\partial l_z^*}{\partial \beta_k} = \sum_{i=1}^N \frac{1}{\pi_{z_i}} (\delta_{kz_i} \phi_{z_i} - \delta_{k(z_i-1)} \phi_{z_i-1}), \text{ where}$$

$$\pi_{z_i} = \Phi(\gamma_{z_i}) - \Phi(\gamma_{z_i-1}), \quad \Phi(\gamma_{z_i}) = \int_{-\infty}^{\gamma_{z_i}} \phi(\gamma) d\gamma,$$

$$\phi(\gamma_{z_i}) = \phi_{z_i} = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\gamma_{z_i}^2}{2}\right\},$$

$$\gamma_{z_i} = \beta_{z_i} + \boldsymbol{\xi}' \mathbf{x}_i, \quad (\phi_0 = \phi_K = \gamma_0 \phi_0 = \gamma_K \phi_K = 0),$$

$$\frac{\partial l_z^*}{\partial \boldsymbol{\xi}} = \sum_{i=1}^N \frac{\mathbf{x}_i}{\pi_{z_i}} (\phi_{z_i} - \phi_{z_i-1}),$$

$$\frac{\partial^2 l_z^*}{\partial \beta_k \partial \beta_l} = \sum_{i=1}^N \left\{ \frac{1}{\pi_{z_i}} (-\delta_{klz_i} \gamma_{z_i} \phi_{z_i} + \delta_{kl(z_i-1)} \gamma_{z_i-1} \phi_{z_i-1}) \right. \\ \left. - \frac{1}{\pi_{z_i}^2} (\delta_{kz_i} \phi_{z_i} - \delta_{k(z_i-1)} \phi_{z_i-1}) (\delta_{lz_i} \phi_{z_i} - \delta_{l(z_i-1)} \phi_{z_i-1}) \right\},$$

where  $\delta_{klz_i} = \delta_{kl} \delta_{lz_i}$ ,

$$\frac{\partial^2 l_z^*}{\partial \boldsymbol{\xi} \partial \beta_k} = \sum_{i=1}^N \mathbf{x}_i \left\{ \frac{1}{\pi_{z_i}} (-\delta_{kz_i} \gamma_{z_i} \phi_{z_i} + \delta_{k(z_i-1)} \gamma_{z_i-1} \phi_{z_i-1}) \right. \\ \left. - \frac{1}{\pi_{z_i}^2} (\delta_{kz_i} \phi_{z_i} - \delta_{k(z_i-1)} \phi_{z_i-1}) (\phi_{z_i} - \phi_{z_i-1}) \right\},$$

$$\frac{\partial^2 l_z^*}{\partial \boldsymbol{\xi} \partial \boldsymbol{\xi}'} = \sum_{i=1}^N \mathbf{x}_i \mathbf{x}_i' \left\{ \frac{1}{\pi_{z_i}} (-\gamma_{z_i} \phi_{z_i} + \gamma_{z_i-1} \phi_{z_i-1}) - \frac{1}{\pi_{z_i}^2} (\phi_{z_i} - \phi_{z_i-1})^2 \right\},$$

$$\begin{aligned}
-N(\mathbf{I}_z)_{\beta_k \beta_l} &= N(\boldsymbol{\Lambda}_z)_{\beta_k \beta_l} = \mathbf{E}_X \mathbf{E}_{Z|X} \left( \frac{\partial^2 l_z^*}{\partial \beta_k \partial \beta_l} \right) \\
&= N \mathbf{E}_X \left( -\delta_{kl} \gamma_k \phi_k + \delta_{kl} \gamma_k \phi_k - \frac{1}{\pi_k} \delta_{kl} \phi_k^2 + \frac{1}{\pi_l} \delta_{(k+1)l} \phi_k \phi_l \right. \\
&\quad \left. + \frac{1}{\pi_k} \delta_{(l+1)k} \phi_k \phi_l - \frac{1}{\pi_{k+1}} \delta_{kl} \phi_k^2 \right) \\
&= N \mathbf{E}_X \left\{ -\delta_{kl} \phi_k^2 \left( \frac{1}{\pi_k} + \frac{1}{\pi_{k+1}} \right) + \phi_k \phi_l \left( \frac{\delta_{(k+1)l}}{\pi_l} + \frac{\delta_{(l+1)k}}{\pi_k} \right) \right\},
\end{aligned}$$

where

$$\mathbf{E}_X \{\cdot\} = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \{\cdot\} p(\mathbf{x}) d\mathbf{x},$$

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{r/2} |\boldsymbol{\Sigma}|^{1/2}} \exp \left\{ -\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right\},$$

$$\begin{aligned}
-N(\mathbf{I}_z)_{\xi \beta_k} &= N(\boldsymbol{\Lambda}_z)_{\xi \beta_k} = \mathbf{E}_X \mathbf{E}_{Z|X} \left( \frac{\partial^2 l_z^*}{\partial \xi \partial \beta_k} \right) \\
&= N \mathbf{E}_X \left\{ \mathbf{x} \left( -\frac{\phi_k^2}{\pi_k} + \frac{1}{\pi_k} \phi_k \phi_{k-1} + \frac{1}{\pi_{k+1}} \phi_{k+1} \phi_k - \frac{\phi_k^2}{\pi_{k+1}} \right) \right\}, \\
-N(\mathbf{I}_z)_{\xi \xi'} &= N(\boldsymbol{\Lambda}_z)_{\xi \xi'} = \mathbf{E}_X \mathbf{E}_{Z|X} \left( \frac{\partial^2 l_z^*}{\partial \xi \partial \xi'} \right) = N \mathbf{E}_X \left\{ \mathbf{x} \mathbf{x}' \sum_{a=1}^K \frac{-1}{\pi_a} (\phi_a - \phi_{a-1})^2 \right\},
\end{aligned}$$

that is,

$$(\boldsymbol{\Lambda}_z)_{\beta_k \beta_l} = \mathbf{E}_X \left\{ -\delta_{kl} \phi_k^2 \left( \frac{1}{\pi_k} + \frac{1}{\pi_{k+1}} \right) + \phi_k \phi_l \left( \frac{\delta_{(k+1)l}}{\pi_l} + \frac{\delta_{(l+1)k}}{\pi_k} \right) \right\},$$

$$(\boldsymbol{\Lambda}_z)_{\xi \beta_k} = \mathbf{E}_X \left[ \mathbf{x} \left\{ -\frac{1}{\pi_k} \phi_k (\phi_k - \phi_{k-1}) + \frac{1}{\pi_{k+1}} \phi_k (\phi_{k+1} - \phi_k) \right\} \right],$$

$$(\boldsymbol{\Lambda}_z)_{\xi \xi'} = \mathbf{E}_X \left\{ \mathbf{x} \mathbf{x}' \sum_{a=1}^K \frac{-1}{\pi_a} (\phi_a - \phi_{a-1})^2 \right\}.$$

## 1.2 Expectations of the products of the log likelihood derivatives

Recall  $\mathbf{L} = \mathbf{\Lambda} + \mathbf{M}$  and  $\bar{l} = N^{-1}l^*$ . Let  $\mathbf{M} = \text{Bdiag}(\mathbf{M}_x, \mathbf{M}_z)$ .

### (1) $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$

$$(1-1) \quad \mathbf{E}(N\mathbf{I}_0^{(2)}) = N \mathbf{E} \left\{ \mathbf{v}(\mathbf{M})' \otimes \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'}, \left( \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'} \right)^{\langle 2 \rangle} \right\} \quad (\text{for } \alpha_1)$$

The nonzero results of

$$N \mathbf{E} \left( \frac{\partial^2 \bar{l}}{\partial (\boldsymbol{\theta}_x)_A \partial (\boldsymbol{\theta}_x)_B} \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_x)_C} \right) = \mathbf{E}_x \left( \frac{\partial^2 \ln p(\mathbf{x})}{\partial (\boldsymbol{\theta}_x)_A \partial (\boldsymbol{\theta}_x)_B} \frac{\partial \ln p(\mathbf{x})}{\partial (\boldsymbol{\theta}_x)_C} \right), \text{ where}$$

$(\cdot)_A$  denotes the  $A$ -th element of the vector in parentheses, are

$$\mathbf{E}_x \left( \frac{\partial^2 \ln p(\mathbf{x})}{\partial \sigma_{ab} \partial \boldsymbol{\mu}} \frac{\partial \ln p(\mathbf{x})}{\partial \boldsymbol{\mu}'} \right) = -\frac{2 - \delta_{ab}}{2} \{ (\boldsymbol{\Sigma}^{-1})_{.a} (\boldsymbol{\Sigma}^{-1})_{.b} + (\boldsymbol{\Sigma}^{-1})_{.b} (\boldsymbol{\Sigma}^{-1})_{.a} \},$$

$$\mathbf{E}_x \left( \frac{\partial^2 \ln p(\mathbf{x})}{\partial \sigma_{ab} \partial \sigma_{cd}} \frac{\partial \ln p(\mathbf{x})}{\partial \sigma_{ef}} \right) = \frac{1}{8} (2 - \delta_{ab})(2 - \delta_{cd})(2 - \delta_{ef}) \left\{ -\sum \sigma^{ac} \sigma^{de} \sigma^{fb} \right\}.$$

$$(1-2) \quad \mathbf{E}(N^2 \mathbf{I}_0^{(1)\langle 3 \rangle}) \quad (\text{for } \alpha_3)$$

The nonzero results of  $\mathbf{E}_x \left\{ \left( \frac{\partial p(\mathbf{x})}{\partial \boldsymbol{\theta}_x} \right)^{\langle 3 \rangle} \right\}$  are

$$\mathbf{E}_x \left( \frac{\partial \ln p(\mathbf{x})}{\partial \boldsymbol{\mu}} \frac{\partial \ln p(\mathbf{x})}{\partial \boldsymbol{\mu}'} \frac{\partial \ln p(\mathbf{x})}{\partial \sigma_{ab}} \right) = \frac{2 - \delta_{ab}}{2} \{ (\boldsymbol{\Sigma}^{-1})_{.a} (\boldsymbol{\Sigma}^{-1})_{.b} + (\boldsymbol{\Sigma}^{-1})_{.b} (\boldsymbol{\Sigma}^{-1})_{.a} \},$$

$$\begin{aligned}
\mathbf{E}_X \left( \frac{\partial \ln p(\mathbf{x})}{\partial \sigma_{ab}} \frac{\partial \ln p(\mathbf{x})}{\partial \sigma_{cd}} \frac{\partial \ln p(\mathbf{x})}{\partial \sigma_{ef}} \right) &= \frac{1}{8} (2 - \delta_{ab})(2 - \delta_{cd})(2 - \delta_{ef}) \\
&\times (2\sigma^{ab}\sigma^{cd}\sigma^{ef} - \sigma^{ab} \sum^3 \sigma^{cd}\sigma^{ef} - \sigma^{cd} \sum^3 \sigma^{ab}\sigma^{ef} - \sigma^{ef} \sum^3 \sigma^{ab}\sigma^{cd} \\
&\quad + \sum^{15} \sigma^{ab}\sigma^{cd}\sigma^{ef}) \\
&= \frac{1}{8} (2 - \delta_{ab})(2 - \delta_{cd})(2 - \delta_{ef}) \sum^8 \sigma^{ac}\sigma^{de}\sigma^{fb}.
\end{aligned}$$

(1-3)  $\mathbf{E}(N^2 \mathbf{I}_0^{(1)<2>} \otimes \mathbf{I}_0^{(2)})$  (for  $\alpha_3$ )

$$\begin{aligned}
&N^2 \mathbf{E} \left( \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_x)_A} \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_x)_B} \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_x)_C} \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_x)_D} \right) \\
&= \sum^3 \mathbf{E}_X \left( \frac{\partial \ln p(\mathbf{x})}{\partial (\boldsymbol{\theta}_x)_A} \frac{\partial \ln p(\mathbf{x})}{\partial (\boldsymbol{\theta}_x)_B} \right) \mathbf{E}_X \left( \frac{\partial \ln p(\mathbf{x})}{\partial (\boldsymbol{\theta}_x)_C} \frac{\partial \ln p(\mathbf{x})}{\partial (\boldsymbol{\theta}_x)_D} \right) + O(N^{-1}), \\
&N^2 \mathbf{E} \left[ \left\{ \frac{\partial^2 \bar{l}}{\partial (\boldsymbol{\theta}_x)_A \partial (\boldsymbol{\theta}_x)_B} - \mathbf{E} \left( \frac{\partial^2 \bar{l}}{\partial (\boldsymbol{\theta}_x)_A \partial (\boldsymbol{\theta}_x)_B} \right) \right\} \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_x)_C} \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_x)_D} \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_x)_E} \right] \\
&= \sum^3 \mathbf{E}_X \left( \frac{\partial^2 \ln p(\mathbf{x})}{\partial (\boldsymbol{\theta}_x)_A \partial (\boldsymbol{\theta}_x)_B} \frac{\partial \ln p(\mathbf{x})}{\partial (\boldsymbol{\theta}_x)_C} \right) \mathbf{E}_X \left( \frac{\partial \ln p(\mathbf{x})}{\partial (\boldsymbol{\theta}_x)_D} \frac{\partial \ln p(\mathbf{x})}{\partial (\boldsymbol{\theta}_x)_E} \right) + O(N^{-1}).
\end{aligned}$$

(1-4)  $\mathbf{E}(N^2 \mathbf{I}_0^{(2)} \mathbf{I}_0^{(2)})$  (for  $\alpha_{\Delta 2}$ )

Define  $\mathbf{E}\{X - \mathbf{E}(\cdot)\}$  as  $\mathbf{E}\{X - \mathbf{E}(X)\}$ . In

$$N^2 \mathbf{E} \left[ \left\{ \frac{\partial^2 \bar{l}}{\partial (\boldsymbol{\theta}_x)_A \partial (\boldsymbol{\theta}_x)_B} - \mathbf{E}(\cdot) \right\} \left\{ \frac{\partial^2 \bar{l}}{\partial (\boldsymbol{\theta}_x)_C \partial (\boldsymbol{\theta}_x)_D} - \mathbf{E}(\cdot) \right\} \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_x)_E} \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_x)_F} \right],$$

we derive the nonzero results of

$$\begin{aligned}
& \mathbf{E}_X \left[ \left\{ \frac{\partial^2 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A \partial(\boldsymbol{\theta}_x)_B} - \mathbf{E}_X(\cdot) \right\} \left\{ \frac{\partial^2 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_C \partial(\boldsymbol{\theta}_x)_D} - \mathbf{E}_X(\cdot) \right\} \right] \\
&= \mathbf{E}_X \{ [(\mathbf{L}_x)_{AB} - (\boldsymbol{\Lambda}_x)_{AB}] [(\mathbf{L}_x)_{CD} - (\boldsymbol{\Lambda}_x)_{CD}] \} \\
&= \mathbf{E}_X \{ [(\mathbf{L}_x)_{AB} - (\boldsymbol{\Lambda}_x)_{AB}] (\mathbf{L}_x)_{CD} \},
\end{aligned}$$

where  $\mathbf{L}_x \equiv \frac{\partial^2 \ln p(\mathbf{x})}{\partial \boldsymbol{\theta}_x \partial \boldsymbol{\theta}_x'}$  ( $\mathbf{L} = \text{Bdiag}(\mathbf{L}_x, \mathbf{L}_z)$ ),

$$\begin{aligned}
& \mathbf{E}_X \left[ \left\{ \frac{\partial^2 \ln p(\mathbf{x})}{\partial \sigma_{ab} \partial \mu_c} - \mathbf{E}_X(\cdot) \right\} \left\{ \frac{\partial^2 \ln p(\mathbf{x})}{\partial \sigma_{de} \partial \mu_f} - \mathbf{E}_X(\cdot) \right\} \right] \\
&= \mathbf{E}_X \left( \frac{\partial^2 \ln p(\mathbf{x})}{\partial \sigma_{ab} \partial \mu_c} \frac{\partial^2 \ln p(\mathbf{x})}{\partial \sigma_{de} \partial \mu_f} \right) = \frac{1}{4} (2 - \delta_{ab})(2 - \delta_{de}) \sum \sigma^{ac} \sigma^{bd} \sigma^{ef}, \\
& \mathbf{E}_X \left[ \left\{ \frac{\partial^2 \ln p(\mathbf{x})}{\partial \sigma_{ab} \partial \sigma_{cd}} - \mathbf{E}_X(\cdot) \right\} \left\{ \frac{\partial^2 \ln p(\mathbf{x})}{\partial \sigma_{ef} \partial \sigma_{gh}} - \mathbf{E}_X(\cdot) \right\} \right] \\
&= \frac{1}{16} (2 - \delta_{ab})(2 - \delta_{cd})(2 - \delta_{ef})(2 - \delta_{gh}) \\
& \times \mathbf{E}_X \left[ \left\{ 2\sigma^{ac} \sigma^{db} + 2\sigma^{ad} \sigma^{cb} - \sum \sigma^{ac} \{ \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) \}_d \{ \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) \}_b \right\} \right. \\
& \quad \left. \times \left\{ 2\sigma^{eg} \sigma^{hf} + 2\sigma^{eh} \sigma^{gf} - \sum \sigma^{eg} \{ \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) \}_h \{ \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) \}_f \right\} \right] \\
&= \frac{1}{16} (2 - \delta_{ab})(2 - \delta_{cd})(2 - \delta_{ef})(2 - \delta_{gh}) \sum \sigma^{ac} \sigma^{eg} (\sigma^{dh} \sigma^{bf} + \sigma^{df} \sigma^{bh}).
\end{aligned}$$

(1-5)  $\mathbf{E}(N^2 \mathbf{I}_0^{(1)} \mathbf{I}_0^{(2)'})$  (for  $\alpha_{\Delta 2}$ )

For  $\mathbf{E}(N^2 \mathbf{I}_0^{(1)<3>})$ , see (1-2). The nonzero results of

$$\begin{aligned} & \mathbf{E}_X \left[ \left\{ (\mathbf{L}_x)_{AB} - (\mathbf{\Lambda}_x)_{AB} \right\} \frac{\partial \ln p(\mathbf{x})}{\partial (\boldsymbol{\theta}_x)_C} \frac{\partial \ln p(\mathbf{x})}{\partial (\boldsymbol{\theta}_x)_D} \right] \\ &= \mathbf{E}_X \left\{ (\mathbf{L}_x)_{AB} \frac{\partial \ln p(\mathbf{x})}{\partial (\boldsymbol{\theta}_x)_C} \frac{\partial \ln p(\mathbf{x})}{\partial (\boldsymbol{\theta}_x)_D} \right\} + (\mathbf{\Lambda}_x)_{AB} (\mathbf{\Lambda}_x)_{CD} \end{aligned}$$

are

$$\mathbf{E}_X \left\{ \frac{\partial^2 \ln p(\mathbf{x})}{\partial \sigma_{ab} \partial \mu_c} \frac{\partial \ln p(\mathbf{x})}{\partial \sigma_{de}} \frac{\partial \ln p(\mathbf{x})}{\partial \mu_f} \right\} = -\frac{1}{4} (2 - \delta_{ab})(2 - \delta_{de}) \sum \sigma^{ca} \sigma^{bd} \sigma^{ef},$$

$$\begin{aligned} & \mathbf{E}_X \left[ \left\{ \frac{\partial^2 \ln p(\mathbf{x})}{\partial \sigma_{ab} \partial \sigma_{cd}} - \mathbf{E}_X(\cdot) \right\} \frac{\partial \ln p(\mathbf{x})}{\partial \mu_e} \frac{\partial \ln p(\mathbf{x})}{\partial \mu_f} \right] \\ &= -\frac{1}{4} (2 - \delta_{ab})(2 - \delta_{cd}) \sum \sigma^{ac} (\sigma^{de} \sigma^{bf} + \sigma^{df} \sigma^{be}), \end{aligned}$$

$$\begin{aligned} & \mathbf{E}_X \left[ \left\{ \frac{\partial^2 \ln p(\mathbf{x})}{\partial \sigma_{ab} \partial \sigma_{cd}} - \mathbf{E}_X(\cdot) \right\} \frac{\partial \ln p(\mathbf{x})}{\partial \sigma_{ef}} \frac{\partial \ln p(\mathbf{x})}{\partial \sigma_{gh}} \right] \\ &= -\frac{1}{16} (2 - \delta_{ab})(2 - \delta_{cd})(2 - \delta_{ef})(2 - \delta_{gh}) \sum \sum \sigma^{ac} \sigma^{de} \sigma^{fg} \sigma^{hb}. \end{aligned}$$

(1-6)  $\mathbf{E}(N^2 \mathbf{I}_0^{(1)} \mathbf{I}_0^{(3)})$  (for  $\alpha_{\Delta 2}$ )

For  $\mathbf{E}(N^2 \mathbf{I}_0^{(1)<4>})$ ,

$$\begin{aligned} & \mathbf{E} \left[ N^2 \left\{ \frac{\partial^2 \bar{l}}{\partial (\boldsymbol{\theta}_x)_A \partial (\boldsymbol{\theta}_x)_B} - \mathbf{E}(\cdot) \right\} \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_x)_C} \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_x)_D} \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_x)_E} \right], \text{ and} \\ & \mathbf{E} \left[ N^2 \left\{ \frac{\partial^2 \bar{l}}{\partial (\boldsymbol{\theta}_x)_A \partial (\boldsymbol{\theta}_x)_B} - \mathbf{E}(\cdot) \right\} \left\{ \frac{\partial^2 \bar{l}}{\partial (\boldsymbol{\theta}_x)_C \partial (\boldsymbol{\theta}_x)_D} - \mathbf{E}(\cdot) \right\} \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_x)_E} \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_x)_F} \right], \end{aligned}$$

see (1-3) and (1-4). The remaining results are

$$\begin{aligned} & \mathbf{E}_X \left[ \left\{ \frac{\partial^3 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A \partial(\boldsymbol{\theta}_x)_B \partial(\boldsymbol{\theta}_x)_C} - \mathbf{E}_X(\cdot) \right\} \frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_D} \right] \\ &= \mathbf{E}_X \left\{ \frac{\partial^3 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A \partial(\boldsymbol{\theta}_x)_B \partial(\boldsymbol{\theta}_x)_C} \frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_D} \right\} \end{aligned}$$

in

$$\mathbf{E} \left[ N^2 \left\{ \frac{\partial^3 \bar{l}}{\partial(\boldsymbol{\theta}_x)_A \partial(\boldsymbol{\theta}_x)_B \partial(\boldsymbol{\theta}_x)_C} - \mathbf{E}(\cdot) \right\} \frac{\partial \bar{l}}{\partial(\boldsymbol{\theta}_x)_D} \frac{\partial \bar{l}}{\partial(\boldsymbol{\theta}_x)_E} \frac{\partial \bar{l}}{\partial(\boldsymbol{\theta}_x)_F} \right]$$

First, we obtain the nonzero  $\frac{\partial^3 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A \partial(\boldsymbol{\theta}_x)_B \partial(\boldsymbol{\theta}_x)_C}$  as

$$\frac{\partial^3 \ln p(\mathbf{x})}{\partial \sigma_{ab} \partial \boldsymbol{\mu} \partial \boldsymbol{\mu}'} = \frac{2 - \delta_{ab}}{2} \{ (\boldsymbol{\Sigma}^{-1})_{.a} (\boldsymbol{\Sigma}^{-1})_{.b} + (\boldsymbol{\Sigma}^{-1})_{.b} (\boldsymbol{\Sigma}^{-1})_{.a} \},$$

$$\begin{aligned} \frac{\partial^3 \ln p(\mathbf{x})}{\partial \sigma_{ab} \partial \sigma_{cd} \partial \boldsymbol{\mu}} &= \frac{1}{4} (2 - \delta_{ab})(2 - \delta_{cd}) \sum^4 \sigma^{ac} [ \{ \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) \}_b (\boldsymbol{\Sigma}^{-1})_{.d} \\ &\quad + \{ \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) \}_d (\boldsymbol{\Sigma}^{-1})_{.b} ], \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^3 \ln p(\mathbf{x})}{\partial \sigma_{ab} \partial \sigma_{cd} \partial \sigma_{ef}} &= \frac{1}{8} (2 - \delta_{ab})(2 - \delta_{cd})(2 - \delta_{ef}) \\ &\quad \times \left[ - \sum^8 \sigma^{ac} \sigma^{de} \sigma^{fb} + \sum^{24} \sigma^{ac} \sigma^{de} \{ \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) \}_f \{ \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) \}_b \right], \end{aligned}$$

which yield the nonzero  $\mathbf{E}_X \left\{ \frac{\partial^3 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A \partial(\boldsymbol{\theta}_x)_B \partial(\boldsymbol{\theta}_x)_C} \frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_D} \right\}$  as

$$\begin{aligned} & \mathbf{E}_X \left\{ \frac{\partial^3 \ln p(\mathbf{x})}{\partial \sigma_{ab} \partial \sigma_{cd} \partial \boldsymbol{\mu}} \frac{\partial \ln p(\mathbf{x})}{\partial \boldsymbol{\mu}'} \right\} \\ &= \frac{1}{4} (2 - \delta_{ab})(2 - \delta_{cd}) \sum^4 \sigma^{ac} \{ (\boldsymbol{\Sigma}^{-1})_{.d} (\boldsymbol{\Sigma}^{-1})_{.b} + (\boldsymbol{\Sigma}^{-1})_{.b} (\boldsymbol{\Sigma}^{-1})_{.d} \}, \end{aligned}$$

$$\mathbf{E}_X \left\{ \frac{\partial^3 \ln p(\mathbf{x})}{\partial \sigma_{ab} \partial \sigma_{cd} \partial \sigma_{ef}} \frac{\partial \ln p(\mathbf{x})}{\partial \sigma_{gh}} \right\} = \frac{1}{16} (2 - \delta_{ab})(2 - \delta_{cd})(2 - \delta_{ef})(2 - \delta_{gh}) \\ \times \sum_{24} \sigma^{ac} \sigma^{de} (\sigma^{fg} \sigma^{hb} + \sigma^{fh} \sigma^{gb}).$$

$$(1-7) \quad \mathbf{E}(N^3 \mathbf{I}_0^{(1)\langle 4 \rangle}) - N \sum^3 \{ \mathbf{E}(N \mathbf{I}_0^{(1)\langle 2 \rangle}) \}^{\langle 2 \rangle} \quad (\text{for } \alpha_4)$$

$$\frac{N^2 - N}{N} \sum^3 \mathbf{E}_X \left\{ \frac{\partial \ln p(\mathbf{x})}{\partial (\boldsymbol{\theta}_x)_A} \frac{\partial \ln p(\mathbf{x})}{\partial (\boldsymbol{\theta}_x)_B} \right\} \mathbf{E}_X \left\{ \frac{\partial \ln p(\mathbf{x})}{\partial (\boldsymbol{\theta}_x)_C} \frac{\partial \ln p(\mathbf{x})}{\partial (\boldsymbol{\theta}_x)_D} \right\} \\ + \mathbf{E}_X \left\{ \frac{\partial \ln p(\mathbf{x})}{\partial (\boldsymbol{\theta}_x)_A} \frac{\partial \ln p(\mathbf{x})}{\partial (\boldsymbol{\theta}_x)_B} \frac{\partial \ln p(\mathbf{x})}{\partial (\boldsymbol{\theta}_x)_C} \frac{\partial \ln p(\mathbf{x})}{\partial (\boldsymbol{\theta}_x)_D} \right\} - N \sum^3 (\boldsymbol{\Lambda}_x)_{AB} (\boldsymbol{\Lambda}_x)_{CD} \\ = \mathbf{E}_X \left\{ \frac{\partial \ln p(\mathbf{x})}{\partial (\boldsymbol{\theta}_x)_A} \frac{\partial \ln p(\mathbf{x})}{\partial (\boldsymbol{\theta}_x)_B} \frac{\partial \ln p(\mathbf{x})}{\partial (\boldsymbol{\theta}_x)_C} \frac{\partial \ln p(\mathbf{x})}{\partial (\boldsymbol{\theta}_x)_D} \right\} - \sum^3 (\boldsymbol{\Lambda}_x)_{AB} (\boldsymbol{\Lambda}_x)_{CD}$$

(the fourth multivariate cumulant of  $\partial \ln p(\mathbf{x}) / \partial (\boldsymbol{\theta}_x)_A$ 's).

The nonzero expectations required for the fourth cumulant are

$$\mathbf{E}_X \left\{ \frac{\partial \ln p(\mathbf{x})}{\partial \sigma_{ab}} \frac{\partial \ln p(\mathbf{x})}{\partial \sigma_{cd}} \frac{\partial \ln p(\mathbf{x})}{\partial \mu_e} \frac{\partial \ln p(\mathbf{x})}{\partial \mu_f} \right\} \\ = \frac{1}{4} (2 - \delta_{ab})(2 - \delta_{cd}) \left\{ \sigma^{ef} (\sigma^{ac} \sigma^{bd} + \sigma^{ad} \sigma^{bc}) + \sum^8 \sigma^{ae} \sigma^{cf} \sigma^{bd} \right\},$$

$$\mathbf{E}_X \left\{ \frac{\partial \ln p(\mathbf{x})}{\partial \sigma_{ab}} \frac{\partial \ln p(\mathbf{x})}{\partial \sigma_{cd}} \frac{\partial \ln p(\mathbf{x})}{\partial \sigma_{ef}} \frac{\partial \ln p(\mathbf{x})}{\partial \sigma_{gh}} \right\} \\ = \frac{1}{16} (2 - \delta_{ab})(2 - \delta_{cd})(2 - \delta_{ef})(2 - \delta_{gh})$$

$$\begin{aligned}
& \times \left\{ -3\sigma^{ab}\sigma^{cd}\sigma^{ef}\sigma^{gh} + \sum^6 \sigma^{ab}\sigma^{cd} (\sigma^{ef}\sigma^{gh} + \sigma^{eg}\sigma^{fh} + \sigma^{eh}\sigma^{fg}) \right. \\
& \quad \left. - \sum^4 \sigma^{ab} \sum^{15} \sigma^{cd}\sigma^{ef}\sigma^{gh} + \sum^{105} \sigma^{ab}\sigma^{cd}\sigma^{ef}\sigma^{gh} \right\} \\
& = \frac{1}{16} (2 - \delta_{ab})(2 - \delta_{cd})(2 - \delta_{ef})(2 - \delta_{gh}) \left( \sum^{6 \times 4 \times 2} \sigma^{ac}\sigma^{de}\sigma^{fg}\sigma^{hb} \right. \\
& \quad \left. + \sum_{(ab,cd,ef,gh)}^3 \sum_{(c,d)}^2 \sigma^{ac}\sigma^{bd} \sum_{(g,h)}^2 \sigma^{eg}\sigma^{fh} \right) \\
& \text{(note that } \mathbf{E}_x \left\{ \frac{\partial \ln p(\mathbf{x})}{\partial \mu_a} \frac{\partial \ln p(\mathbf{x})}{\partial \mu_b} \frac{\partial \ln p(\mathbf{x})}{\partial \mu_c} \frac{\partial \ln p(\mathbf{x})}{\partial \mu_d} \right\} = \sum^3 \sigma^{ab}\sigma^{cd} \text{ is} \\
& \text{nonzero, but the corresponding cumulant is zero).}
\end{aligned}$$

$$(1-8) \quad \mathbf{E}(N^3 \mathbf{I}_0^{(1)\langle 3 \rangle} \otimes \mathbf{I}_0^{(2)}) \quad (\text{for } \alpha_4)$$

The expectations are expressed as  $\sum^{10} (\cdot)$ , where each  $(\cdot)$  is the product of two expectations, which was shown in (1-2), (1-3), and (1-5).

$$(1-9) \quad \mathbf{E}(N^3 \mathbf{I}_0^{(1)\langle 2 \rangle} \otimes \mathbf{I}_0^{(2)\langle 2 \rangle}) \quad (\text{for } \alpha_4)$$

The results are expressed as  $\sum^{15} (\cdot)$ , where each  $(\cdot)$  is the product of three expectations, which was shown in (1-1) and (1-4).

$$(1-10) \quad \mathbf{E}(N^3 \mathbf{I}_0^{(1)\langle 3 \rangle} \otimes \mathbf{I}_0^{(3)}) \quad (\text{for } \alpha_4)$$

The results are expressed as  $\sum^{15} (\cdot)$ , where each  $(\cdot)$  is the product of three expectations, which was shown in (1-6).

$$(1-11) \quad \mathbf{E} \left( \frac{\partial^3 \bar{l}}{\partial (\boldsymbol{\theta}_x)_A \partial (\boldsymbol{\theta}_x)_B \partial (\boldsymbol{\theta}_x)_C} \right) \quad (\text{for } \alpha_1 \text{ and } \alpha_3)$$

The results are given by the Bartlett identity as follows:

$$\begin{aligned} \mathbb{E} \left( \frac{\partial^3 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A \partial(\boldsymbol{\theta}_x)_B \partial(\boldsymbol{\theta}_x)_C} \right) &= -\sum^3 \mathbb{E} \left( \frac{\partial^2 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A \partial(\boldsymbol{\theta}_x)_B} \frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_C} \right) \\ &\quad - \mathbb{E} \left( \frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A} \frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_B} \frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_C} \right). \end{aligned}$$

$$(1-12) \quad \mathbb{E} \left( \frac{\partial^4 \bar{l}}{\partial(\boldsymbol{\theta}_x)_A \partial(\boldsymbol{\theta}_x)_B \partial(\boldsymbol{\theta}_x)_C \partial(\boldsymbol{\theta}_x)_D} \right) \quad (\text{for } \alpha_{\Delta 2} \text{ and } \alpha_4)$$

As in (1-11), we obtain

$$\begin{aligned} \mathbb{E} \left( \frac{\partial^4 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A \partial(\boldsymbol{\theta}_x)_B \partial(\boldsymbol{\theta}_x)_C \partial(\boldsymbol{\theta}_x)_D} \right) &= -\sum^4 \mathbb{E} \left( \frac{\partial^3 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A \partial(\boldsymbol{\theta}_x)_B \partial(\boldsymbol{\theta}_x)_C} \frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_D} \right) \\ &\quad - \sum^3 \mathbb{E} \left( \frac{\partial^2 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A \partial(\boldsymbol{\theta}_x)_B} \frac{\partial^2 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_C \partial(\boldsymbol{\theta}_x)_D} \right) - \sum^6 \mathbb{E} \left( \frac{\partial^2 \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A \partial(\boldsymbol{\theta}_x)_B} \right. \\ &\quad \left. \times \frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_C} \frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_D} \right) - \mathbb{E} \left( \frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_A} \frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_B} \frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_C} \frac{\partial \ln p(\mathbf{x})}{\partial(\boldsymbol{\theta}_x)_D} \right). \end{aligned}$$

## (2) $\boldsymbol{\beta}$ and $\boldsymbol{\xi}$

$$(2-1) \quad \mathbb{E}(N\mathbf{I}_0^{(2)}) = N \mathbb{E} \left\{ \mathbf{v}(\mathbf{M})' \otimes \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'}, \left( \frac{\partial \bar{l}}{\partial \boldsymbol{\theta}_0'} \right)^{\langle 2 \rangle} \right\} \quad (\text{for } \alpha_1)$$

The results are

$$\begin{aligned}
& \mathbf{E} \left( \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \beta_k \partial \beta_l} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_m} \right) \\
&= \mathbf{E} \left[ \left\{ \delta_{kl} \left( -\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \gamma_k \phi_k - \delta_{kl} \left( \frac{\delta_{zk}}{\pi_k^2} + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} \right) \phi_k^2 \right. \right. \\
&\quad \left. \left. + \left( \frac{\delta_{zk(l+1)}}{\pi_k^2} + \frac{\delta_{z(k+1)l}}{\pi_{k+1}^2} \right) \phi_k \phi_l \right\} \left( \frac{\delta_{zm}}{\pi_m} - \frac{\delta_{z(m+1)}}{\pi_{m+1}} \right) \phi_m \right] \\
&= \mathbf{E}_X \left[ \left\{ \delta_{kl} \left( -\frac{\gamma_k \phi_k}{\pi_k} - \frac{\phi_k^2}{\pi_k^2} \right) + \frac{\delta_{k(l+1)}}{\pi_k^2} \phi_k \phi_{k-1} \right\} (\delta_{mk} \phi_k - \delta_{(m+1)k} \phi_{k-1}) \right. \\
&\quad \left. + \left\{ \delta_{kl} \left( \frac{\gamma_k \phi_k}{\pi_{k+1}} - \frac{\phi_k^2}{\pi_{k+1}^2} \right) + \frac{\delta_{(k+1)l}}{\pi_{k+1}^2} \phi_k \phi_{k+1} \right\} (\delta_{m(k+1)} \phi_{k+1} - \delta_{mk} \phi_k) \right],
\end{aligned}$$

$$\begin{aligned}
& \mathbf{E} \left( \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \beta_k \partial \beta_l} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi} \right) \\
&= \mathbf{E} \left[ \left\{ \delta_{kl} \left( -\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \gamma_k \phi_k - \delta_{kl} \left( \frac{\delta_{zk}}{\pi_k^2} + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} \right) \phi_k^2 \right. \right. \\
&\quad \left. \left. + \left( \frac{\delta_{zk(l+1)}}{\pi_k^2} + \frac{\delta_{z(k+1)l}}{\pi_{k+1}^2} \right) \phi_k \phi_l \right\} \sum_{a=1}^K \mathbf{x} \frac{\delta_{za}}{\pi_a} (\phi_a - \phi_{a-1}) \right] \\
&= \mathbf{E}_X \left( \mathbf{x} \left[ \left\{ \delta_{kl} \left( -\frac{\gamma_k \phi_k}{\pi_k} - \frac{\phi_k^2}{\pi_k^2} \right) + \frac{\delta_{k(l+1)}}{\pi_k^2} \phi_k \phi_{k-1} \right\} (\phi_k - \phi_{k-1}) \right. \right. \\
&\quad \left. \left. + \left\{ \delta_{kl} \left( \frac{\gamma_k \phi_k}{\pi_{k+1}} - \frac{\phi_k^2}{\pi_{k+1}^2} \right) + \frac{\delta_{(k+1)l}}{\pi_{k+1}^2} \phi_k \phi_{k+1} \right\} (\phi_{k+1} - \phi_k) \right] \right),
\end{aligned}$$

$$\begin{aligned}
& \mathbf{E} \left( \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \xi \partial \beta_k} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_l} \right) \\
&= \mathbf{E} \left[ \mathbf{x} \left\{ \left( -\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \gamma_k \phi_k - \frac{\delta_{zk}}{\pi_k^2} \phi_k (\phi_k - \phi_{k-1}) + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} \phi_k (\phi_{k+1} - \phi_k) \right\} \right. \\
&\quad \left. \times \left( \frac{\delta_{zl}}{\pi_l} - \frac{\delta_{z(l+1)}}{\pi_{l+1}} \right) \phi_l \right] \\
&= \mathbf{E}_X \left( \mathbf{x} \left[ \delta_{kl} \left\{ -\frac{\gamma_k \phi_k}{\pi_k} - \frac{1}{\pi_k^2} \phi_k (\phi_k - \phi_{k-1}) - \left( \frac{\gamma_k \phi_k}{\pi_{k+1}} + \frac{1}{\pi_{k+1}^2} \phi_k (\phi_{k+1} - \phi_k) \right) \right\} \phi_k \right. \right. \\
&\quad \left. \left. + \delta_{k(l+1)} \left\{ \frac{\gamma_k \phi_k}{\pi_k} + \frac{1}{\pi_k^2} \phi_k (\phi_k - \phi_{k-1}) \right\} \phi_{k-1} \right. \right. \\
&\quad \left. \left. + \delta_{(k+1)l} \left\{ \frac{\gamma_k \phi_k}{\pi_{k+1}} + \frac{1}{\pi_{k+1}^2} \phi_k (\phi_{k+1} - \phi_k) \right\} \phi_{k+1} \right] \right),
\end{aligned}$$

$$\begin{aligned}
& \mathbf{E} \left( \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \xi \partial \beta_k} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi'} \right) \\
&= \mathbf{E} \left[ \mathbf{x} \mathbf{x}' \left\{ \left( -\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \gamma_k \phi_k - \frac{\delta_{zk}}{\pi_k^2} \phi_k (\phi_k - \phi_{k-1}) + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} \phi_k (\phi_{k+1} - \phi_k) \right\} \right. \\
&\quad \left. \times \sum_{a=1}^K \frac{\delta_{za}}{\pi_a} (\phi_a - \phi_{a-1}) \right] \\
&= \mathbf{E}_X \left( \mathbf{x} \mathbf{x}' \left[ \left\{ -\frac{\gamma_k \phi_k}{\pi_k} - \frac{1}{\pi_k^2} \phi_k (\phi_k - \phi_{k-1}) \right\} (\phi_k - \phi_{k-1}) \right. \right. \\
&\quad \left. \left. + \left\{ \frac{\gamma_k \phi_k}{\pi_{k+1}} + \frac{1}{\pi_{k+1}^2} \phi_k (\phi_{k+1} - \phi_k) \right\} (\phi_{k+1} - \phi_k) \right] \right),
\end{aligned}$$

$$\begin{aligned}
& \mathbb{E} \left( \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \xi \partial \xi'} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_k} \right) \\
&= \mathbb{E} \left[ \mathbf{x} \mathbf{x}' \sum_{m=1}^K \left\{ \frac{1}{\pi_m} (-\gamma_m \phi_m + \gamma_{m-1} \phi_{m-1}) - \frac{1}{\pi_m^2} (\phi_m - \phi_{m-1})^2 \right\} \right. \\
&\quad \left. \times \delta_{zm} \left( \frac{\delta_{zk}}{\pi_k} - \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \phi_k \right] \\
&= \mathbb{E}_X \left[ \mathbf{x} \mathbf{x}' \left\{ \frac{1}{\pi_k} (-\gamma_k \phi_k + \gamma_{k-1} \phi_{k-1}) - \frac{1}{\pi_k^2} (\phi_k - \phi_{k-1})^2 \right. \right. \\
&\quad \left. \left. - \frac{1}{\pi_{k+1}} (-\gamma_{k+1} \phi_{k+1} + \gamma_k \phi_k) + \frac{1}{\pi_{k+1}^2} (\phi_{k+1} - \phi_k)^2 \right\} \phi_k \right],
\end{aligned}$$

$$\begin{aligned}
& \mathbb{E} \left( \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \xi_k \partial \xi_l} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi_m} \right) \\
&= \mathbb{E} \left[ x_k x_l x_m \sum_{a=1}^K \left\{ \frac{1}{\pi_a} (-\gamma_a \phi_a + \gamma_{a-1} \phi_{a-1}) - \frac{1}{\pi_a^2} (\phi_a - \phi_{a-1})^2 \right\} \delta_{za} \right. \\
&\quad \left. \times \sum_{b=1}^K \frac{\delta_{zb}}{\pi_b} (\phi_b - \phi_{b-1}) \right] \\
&= \mathbb{E}_X \left[ x_k x_l x_m \sum_{a=1}^K \left\{ \frac{1}{\pi_a} (-\gamma_a \phi_a + \gamma_{a-1} \phi_{a-1}) - \frac{1}{\pi_a^2} (\phi_a - \phi_{a-1})^2 \right\} (\phi_a - \phi_{a-1}) \right].
\end{aligned}$$

(2-2)  $\mathbb{E}(N^2 \mathbf{1}_0^{(1)<3>})$  (for  $\alpha_3$ )

$$\begin{aligned}
& \mathbb{E} \left( \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_k} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_l} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_m} \right) \\
&= \mathbb{E} \left\{ \left( \frac{\delta_{zk}}{\pi_k} - \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \phi_k \left( \frac{\delta_{zl}}{\pi_l} - \frac{\delta_{z(l+1)}}{\pi_{l+1}} \right) \phi_l \left( \frac{\delta_{zm}}{\pi_m} - \frac{\delta_{z(m+1)}}{\pi_{m+1}} \right) \phi_m \right\}
\end{aligned}$$

$$\begin{aligned}
&= \mathbf{E}_X \left\{ \delta_{klm} \left( \frac{1}{\pi_k^2} - \frac{1}{\pi_{k+1}^2} \right) \phi_k^3 - \sum_{(klm)}^3 \delta_{(k+1)lm} \frac{1}{\pi_{k+1}^2} \phi_k \phi_{k+1}^2 \right. \\
&\quad \left. + \sum_{(klm)}^3 \delta_{(k+1)(l+1)m} \frac{1}{\pi_{k+1}^2} \phi_k^2 \phi_{k+1} \right\}, \\
&\mathbf{E} \left( \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_k} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_l} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi} \right) \\
&= \mathbf{E} \left\{ \mathbf{x} \left( \frac{\delta_{zk}}{\pi_k} - \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \phi_k \left( \frac{\delta_{zl}}{\pi_l} - \frac{\delta_{z(l+1)}}{\pi_{l+1}} \right) \phi_l \sum_{a=1}^K \frac{\delta_{za}}{\pi_a} (\phi_a - \phi_{a-1}) \right\} \\
&= \mathbf{E}_X \left( \mathbf{x} \left[ \delta_{kl} \left\{ \frac{1}{\pi_k^2} \phi_k^2 (\phi_k - \phi_{k-1}) + \frac{1}{\pi_{k+1}^2} \phi_k^2 (\phi_{k+1} - \phi_k) \right\} \right. \right. \\
&\quad \left. \left. - \sum_{(kl)}^2 \delta_{(k+1)l} \frac{1}{\pi_{k+1}^2} \phi_{k+1} \phi_k (\phi_{k+1} - \phi_k) \right] \right), \\
&\mathbf{E} \left( \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_k} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi'} \right) \\
&= \mathbf{E} \left[ \mathbf{x} \mathbf{x}' \left( \frac{\delta_{zk}}{\pi_k} - \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \phi_k \left\{ \sum_{a=1}^K \frac{\delta_{za}}{\pi_a} (\phi_a - \phi_{a-1}) \right\}^2 \right] \\
&= \mathbf{E}_X \left[ \mathbf{x} \mathbf{x}' \left\{ \frac{1}{\pi_k^2} \phi_k (\phi_k - \phi_{k-1})^2 - \frac{1}{\pi_{k+1}^2} \phi_k (\phi_{k+1} - \phi_k)^2 \right\} \right], \\
&\mathbf{E} \left( \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi_k} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi_l} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi_m} \right) \\
&= \mathbf{E} \left[ x_k x_l x_m \left\{ \sum_{a=1}^K \frac{\delta_{za}}{\pi_a} (\phi_a - \phi_{a-1}) \right\}^3 \right] = \mathbf{E}_X \left\{ x_k x_l x_m \sum_{a=1}^K \frac{1}{\pi_a^2} (\phi_a - \phi_{a-1})^3 \right\}.
\end{aligned}$$

$$(2-3) \quad \mathbf{E}(N^2 \mathbf{I}_0^{(1)<2>} \otimes \mathbf{I}_0^{(2)}) \quad (\text{for } \alpha_3)$$

$$\begin{aligned}
& N^2 \mathbf{E} \left( \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_z)_A} \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_z)_B} \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_z)_C} \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_z)_D} \right) \\
&= \sum^3 \mathbf{E} \left( \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_A} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_B} \right) \mathbf{E} \left( \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_C} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_D} \right) \\
&\quad + O(N^{-1})
\end{aligned}$$

(see the expression of  $-\Lambda_z$  in 1.1 (2)),

$$\begin{aligned}
& N^2 \mathbf{E} \left[ \left\{ \frac{\partial^2 \bar{l}}{\partial (\boldsymbol{\theta}_z)_A \partial (\boldsymbol{\theta}_z)_B} - \mathbf{E}(\cdot) \right\} \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_z)_C} \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_z)_D} \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_z)_E} \right] \\
&= \sum^3 \mathbf{E} \left( \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_A \partial (\boldsymbol{\theta}_z)_B} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_C} \right) \mathbf{E} \left( \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_D} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_E} \right) \\
&\quad + O(N^{-1})
\end{aligned}$$

(see (2-1) and the expression of  $-\Lambda_z$  in 1.1 (2)).

(2-4)  $\mathbf{E}(N^2 \mathbf{I}_0^{(2)} \mathbf{I}_0^{(2)'})$  (for  $\alpha_{\Delta 2}$ )

For  $\mathbf{E}(N^2 \mathbf{I}_0^{(1)<2>} \otimes \mathbf{I}_0^{(2)})$ , see (2-3). Recall  $\mathbf{L}_z = \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \boldsymbol{\theta}_z \partial \boldsymbol{\theta}_z'}$ , then the

remaining results are  $\mathbf{E}[\{(\mathbf{L}_z)_{AB} - (\Lambda_z)_{AB}\} \{(\mathbf{L}_z)_{CD} - (\Lambda_z)_{CD}\}]$  in

$$\sum^3 \mathbf{E}[\{(\mathbf{L}_z)_{AB} - (\Lambda_z)_{AB}\} \{(\mathbf{L}_z)_{CD} - (\Lambda_z)_{CD}\}] \mathbf{E} \left( \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_E} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_F} \right).$$

We obtain

$$\begin{aligned}
& \mathbf{E} \left( \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \beta_k \partial \beta_l} \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \beta_m \partial \beta_n} \right) - (\Lambda_z)_{\beta_k \beta_l} (\Lambda_z)_{\beta_m \beta_n} \\
&= \mathbf{E} \left[ \left\{ \delta_{kl} \left( -\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \gamma_k \phi_k - \delta_{kl} \left( \frac{\delta_{zk}}{\pi_k^2} + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} \right) \phi_k^2 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \left( \frac{\delta_{zk(l+1)}}{\pi_k^2} + \frac{\delta_{z(k+1)l}}{\pi_{k+1}^2} \right) \phi_k \phi_l \Big\} \\
& \times \left\{ \delta_{mn} \left( -\frac{\delta_{zm}}{\pi_m} + \frac{\delta_{z(m+1)}}{\pi_{m+1}} \right) \gamma_m \phi_m - \delta_{mn} \left( \frac{\delta_{zm}}{\pi_m^2} + \frac{\delta_{z(m+1)}}{\pi_{m+1}^2} \right) \phi_m^2 \right. \\
& \quad \left. + \left( \frac{\delta_{zm(n+1)}}{\pi_m^2} + \frac{\delta_{z(m+1)n}}{\pi_{m+1}^2} \right) \phi_m \phi_n \right\} - (\Lambda_z)_{\beta_k \beta_l} (\Lambda_z)_{\beta_m \beta_n} \\
& = \mathbf{E}_X \left( \delta_{klmn} \left\{ \left( \gamma_k \phi_k + \frac{\phi_k^2}{\pi_k} \right)^2 \frac{1}{\pi_k} + \left( \gamma_k \phi_k - \frac{\phi_k^2}{\pi_{k+1}} \right)^2 \frac{1}{\pi_{k+1}} \right\} \right. \\
& \quad + \sum_{(kl, mn)}^2 \delta_{kl(m+1)(n+1)} \left\{ - \left( \gamma_k \phi_k + \frac{\phi_k^2}{\pi_k} \right) \left( \gamma_{k-1} \phi_{k-1} - \frac{\phi_{k-1}^2}{\pi_k} \right) \frac{1}{\pi_k} \right\} \\
& \quad + \sum_{(klmn)}^4 \left[ \delta_{klm(n+1)} \left\{ - \left( \gamma_k \phi_k + \frac{\phi_k^2}{\pi_k} \right) \frac{1}{\pi_k^2} \phi_k \phi_{k-1} \right\} \right. \\
& \quad \quad \left. + \delta_{(k+1)(l+1)(m+1)n} \left( \gamma_k \phi_k - \frac{\phi_k^2}{\pi_{k+1}} \right) \frac{1}{\pi_{k+1}^2} \phi_k \phi_{k+1} \right] \\
& \quad + \sum_{(mn)}^2 \delta_{k(l+1)m(n+1)} \frac{1}{\pi_k^3} \phi_k^2 \phi_{k-1}^2 + \sum_{(mn)}^2 \delta_{(k+1)lm(n+1)} \frac{1}{\pi_{k+1}^3} \phi_k^2 \phi_{k+1}^2 \Big) - (\Lambda_z)_{\beta_k \beta_l} (\Lambda_z)_{\beta_m \beta_n}, \\
& \mathbf{E} \left( \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \beta_k \partial \beta_l} \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \xi \partial \beta_m} \right) - (\Lambda_z)_{\beta_k \beta_l} (\Lambda_z)_{\xi \beta_m} \\
& = \mathbf{E} \left( \mathbf{x} \left[ \left\{ \delta_{kl} \left( -\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \gamma_k \phi_k - \delta_{kl} \left( \frac{\delta_{zk}}{\pi_k^2} + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} \right) \phi_k^2 \right. \right. \right. \\
& \quad \left. \left. + \left( \frac{\delta_{zk(l+1)}}{\pi_k^2} + \frac{\delta_{z(k+1)l}}{\pi_{k+1}^2} \right) \phi_k \phi_l \right\} \left\{ \left( -\frac{\delta_{zm}}{\pi_m} + \frac{\delta_{z(m+1)}}{\pi_{m+1}} \right) \gamma_m \phi_m - \frac{\delta_{zm}}{\pi_m^2} \phi_m (\phi_m - \phi_{m-1}) \right. \right. \right. \\
\end{aligned}$$

$$\begin{aligned}
& \left. + \frac{\delta_{z(m+1)}}{\pi_{m+1}^2} \phi_m (\phi_{m+1} - \phi_m) \right\} \Big] - (\Lambda_z)_{\beta_k \beta_l} (\Lambda_z)_{\xi \beta_m} \\
= & \mathbf{E}_X \left( \mathbf{x} \left[ \delta_{klm} \left\{ \left( \gamma_k \phi_k + \frac{\phi_k^2}{\pi_k} \right) \left( \frac{\gamma_m \phi_m}{\pi_m} + \frac{1}{\pi_m^2} \phi_m (\phi_m - \phi_{m-1}) \right) \right. \right. \right. \\
& \quad \left. \left. \left. + \left( \gamma_k \phi_k - \frac{\phi_k^2}{\pi_{k+1}} \right) \left( \frac{\gamma_m \phi_m}{\pi_{m+1}} + \frac{1}{\pi_{m+1}^2} \phi_m (\phi_{m+1} - \phi_m) \right) \right\} \right. \right. \\
& \quad \left. \left. + \delta_{kl(m+1)} \left\{ - \left( \gamma_k \phi_k + \frac{\phi_k^2}{\pi_k} \right) \left( \frac{\gamma_m \phi_m}{\pi_{m+1}} + \frac{1}{\pi_{m+1}^2} \phi_m (\phi_{m+1} - \phi_m) \right) \right\} \right. \right. \\
& \quad \left. \left. + \delta_{(k+1)(l+1)m} \left( -\gamma_k \phi_k + \frac{\phi_k^2}{\pi_{k+1}} \right) \left( \frac{\gamma_m \phi_m}{\pi_m} + \frac{1}{\pi_m^2} \phi_m (\phi_m - \phi_{m-1}) \right) \right. \right. \\
& \quad \left. \left. + \sum_{(kl)}^2 \delta_{k(l+1)m} \frac{-1}{\pi_k} \phi_k \phi_{k-1} \left\{ \frac{\gamma_m \phi_m}{\pi_m} + \frac{1}{\pi_m^2} \phi_m (\phi_m - \phi_{m-1}) \right\} \right. \right. \\
& \quad \left. \left. + \sum_{(kl)}^2 \delta_{k(l+1)(m+1)} \frac{1}{\pi_k} \phi_k \phi_{k-1} \left\{ \frac{\gamma_m \phi_m}{\pi_{m+1}} + \frac{1}{\pi_{m+1}^2} \phi_m (\phi_{m+1} - \phi_m) \right\} \right] \right) - (\Lambda_z)_{\beta_k \beta_l} (\Lambda_z)_{\xi \beta_m}, \\
& \mathbf{E} \left( \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \beta_k \partial \beta_l} \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \xi \partial \xi'} \right) - (\Lambda_z)_{\beta_k \beta_l} (\Lambda_z)_{\xi \xi'}, \\
= & \mathbf{E} \left[ \mathbf{x} \mathbf{x}' \left\{ \delta_{kl} \left( -\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \gamma_k \phi_k - \delta_{kl} \left( \frac{\delta_{zk}}{\pi_k^2} + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} \right) \phi_k^2 \right. \right. \\
& \quad \left. \left. + \left( \frac{\delta_{zk(l+1)}}{\pi_k^2} + \frac{\delta_{z(k+1)l}}{\pi_{k+1}^2} \right) \phi_k \phi_l \right\} \right. \\
& \quad \left. \times \sum_{m=1}^K \left\{ \frac{1}{\pi_m} (-\gamma_m \phi_m + \gamma_{m-1} \phi_{m-1}) - \frac{1}{\pi_m^2} (\phi_m - \phi_{m-1})^2 \right\} \delta_{zm} \right] - (\Lambda_z)_{\beta_k \beta_l} (\Lambda_z)_{\xi \xi'},
\end{aligned}$$

$$\begin{aligned}
&= \mathbf{E}_X \left( \mathbf{x} \mathbf{x}' \left[ \delta_{kl} \left\{ - \left( \gamma_k \phi_k + \frac{\phi_k^2}{\pi_k} \right) \left( \frac{1}{\pi_k} (-\gamma_k \phi_k + \gamma_{k-1} \phi_{k-1}) - \frac{1}{\pi_k^2} (\phi_k - \phi_{k-1})^2 \right) \right. \right. \right. \\
&\quad \left. \left. \left. + \left( \gamma_k \phi_k - \frac{\phi_k^2}{\pi_{k+1}} \right) \left( \frac{1}{\pi_{k+1}} (-\gamma_{k+1} \phi_{k+1} + \gamma_k \phi_k) - \frac{1}{\pi_{k+1}^2} (\phi_{k+1} - \phi_k)^2 \right) \right\} \right. \right. \\
&\quad \left. \left. + \sum_{(kl)}^2 \delta_{k(l+1)} \frac{1}{\pi_k} \phi_k \phi_{k-1} \left\{ \frac{1}{\pi_k} (-\gamma_k \phi_k + \gamma_{k-1} \phi_{k-1}) - \frac{1}{\pi_k^2} (\phi_k - \phi_{k-1})^2 \right\} \right] \right) \\
&\quad - (\Lambda_z)_{\beta_k \beta_l} (\Lambda_z)_{\xi \xi'}, \\
&\quad \mathbf{E} \left( \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \xi \partial \beta_k} \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \xi' \partial \beta_l} \right) - (\Lambda_z)_{\xi \beta_k} (\Lambda_z)_{\beta_l \xi'}, \\
&= \mathbf{E} \left[ \mathbf{x} \mathbf{x}' \left\{ \left( -\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \gamma_k \phi_k - \frac{\delta_{zk}}{\pi_k^2} \phi_k (\phi_k - \phi_{k-1}) + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} \phi_k (\phi_{k+1} - \phi_k) \right\} \right. \\
&\quad \left. \times \left\{ \left( -\frac{\delta_{zl}}{\pi_l} + \frac{\delta_{z(l+1)}}{\pi_{l+1}} \right) \gamma_l \phi_l - \frac{\delta_{zl}}{\pi_l^2} \phi_l (\phi_l - \phi_{l-1}) + \frac{\delta_{z(l+1)}}{\pi_{l+1}^2} \phi_l (\phi_{l+1} - \phi_l) \right\} \right] \\
&\quad - (\Lambda_z)_{\xi \beta_k} (\Lambda_z)_{\beta_l \xi'}, \\
&= \mathbf{E}_X \left( \mathbf{x} \mathbf{x}' \left[ \delta_{kl} \left\{ \frac{1}{\pi_k} \left( \gamma_k \phi_k + \frac{1}{\pi_k} \phi_k (\phi_k - \phi_{k-1}) \right)^2 \right. \right. \right. \\
&\quad \left. \left. \left. + \frac{1}{\pi_{k+1}} \left( \gamma_k \phi_k + \frac{1}{\pi_{k+1}} \phi_k (\phi_{k+1} - \phi_k) \right)^2 \right\} \right. \right. \\
&\quad \left. \left. - \sum_{(kl)}^2 \delta_{k(l+1)} \left\{ \gamma_k \phi_k + \frac{1}{\pi_k} \phi_k (\phi_k - \phi_{k-1}) \right\} \left\{ \frac{1}{\pi_k} \gamma_{k-1} \phi_{k-1} + \frac{1}{\pi_k^2} \phi_{k-1} (\phi_k - \phi_{k-1}) \right\} \right] \right) \\
&\quad - (\Lambda_z)_{\xi \beta_k} (\Lambda_z)_{\beta_l \xi'},
\end{aligned}$$

$$\begin{aligned}
& \mathbf{E} \left( \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \xi_k \partial \beta_l} \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \xi \partial \xi'} \right) - (\boldsymbol{\Lambda}_z)_{\xi_k \beta_l} (\boldsymbol{\Lambda}_z)_{\xi \xi'} \\
&= \mathbf{E} \left[ x_k \mathbf{x} \mathbf{x}' \left\{ \left( -\frac{\delta_{zl}}{\pi_l} + \frac{\delta_{z(l+1)}}{\pi_{l+1}} \right) \gamma_l \phi_l - \frac{\delta_{zl}}{\pi_l^2} \phi_l (\phi_l - \phi_{l-1}) + \frac{\delta_{z(l+1)}}{\pi_{l+1}^2} \phi_l (\phi_{l+1} - \phi_l) \right\} \right. \\
&\quad \times \sum_{m=1}^K \left\{ \frac{1}{\pi_m} (-\gamma_m \phi_m + \gamma_{m-1} \phi_{m-1}) - \frac{1}{\pi_m^2} (\phi_m - \phi_{m-1})^2 \right\} \delta_{zm} \left. \right] - (\boldsymbol{\Lambda}_z)_{\xi_k \beta_l} (\boldsymbol{\Lambda}_z)_{\xi \xi'} \\
&= \mathbf{E}_X \left( x_k \mathbf{x} \mathbf{x}' \left[ - \left\{ \gamma_l \phi_l + \frac{1}{\pi_l} \phi_l (\phi_l - \phi_{l-1}) \right\} \right. \right. \\
&\quad \times \left\{ \frac{1}{\pi_l} (-\gamma_l \phi_l + \gamma_{l-1} \phi_{l-1}) - \frac{1}{\pi_l^2} (\phi_l - \phi_{l-1})^2 \right\} + \left\{ \gamma_l \phi_l + \frac{1}{\pi_{l+1}} \phi_l (\phi_{l+1} - \phi_l) \right\} \\
&\quad \times \left. \left. \left\{ \frac{1}{\pi_{l+1}} (-\gamma_{l+1} \phi_{l+1} + \gamma_l \phi_l) - \frac{1}{\pi_{l+1}^2} (\phi_{l+1} - \phi_l)^2 \right\} \right] \right] - (\boldsymbol{\Lambda}_z)_{\xi_k \beta_l} (\boldsymbol{\Lambda}_z)_{\xi \xi'}, \\
& \mathbf{E} \left( \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \xi_k \partial \xi_l} \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \xi_m \partial \xi_n} \right) - (\boldsymbol{\Lambda}_z)_{\xi_k \xi_l} (\boldsymbol{\Lambda}_z)_{\xi_m \xi_n} \\
&= \mathbf{E} \left( x_k x_l x_m x_n \left[ \sum_{a=1}^K \left\{ \frac{1}{\pi_a} (-\gamma_a \phi_a + \gamma_{a-1} \phi_{a-1}) - \frac{1}{\pi_a^2} (\phi_a - \phi_{a-1})^2 \right\} \delta_{za} \right]^2 \right) \\
&\quad - (\boldsymbol{\Lambda}_z)_{\xi_k \xi_l} (\boldsymbol{\Lambda}_z)_{\xi_m \xi_n} \\
&= \mathbf{E}_X \left[ x_k x_l x_m x_n \sum_{a=1}^K \frac{1}{\pi_a} \left\{ -\gamma_a \phi_a + \gamma_{a-1} \phi_{a-1} - \frac{1}{\pi_a} (\phi_a - \phi_{a-1})^2 \right\}^2 \right] \\
&\quad - (\boldsymbol{\Lambda}_z)_{\xi_k \xi_l} (\boldsymbol{\Lambda}_z)_{\xi_m \xi_n}
\end{aligned}$$

(2-5)  $\mathbf{E}(N^2 \mathbf{1}_0^{(1)} \mathbf{1}_0^{(2)})$  (for  $\alpha_{\Delta 2}$ )

For  $\mathbf{E}(N^2 \mathbf{1}_0^{(1)<3>})$ , see (2-2). The remaining results are

$$\begin{aligned} & \mathbb{E} \left[ \{(\mathbf{L}_z)_{AB} - (\mathbf{\Lambda}_z)_{AB}\} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_C} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_D} \right] \\ &= \mathbb{E}_X \left\{ (\mathbf{L}_z)_{AB} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_C} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_D} \right\} + (\mathbf{\Lambda}_z)_{AB} (\mathbf{\Lambda}_z)_{CD}. \end{aligned}$$

We obtain the first term on the right-hand side of the above equation as follows:

$$\begin{aligned} & \mathbb{E} \left( \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \beta_k \partial \beta_l} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_m} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_n} \right) \\ &= \mathbb{E} \left\{ \left[ \delta_{kl} \left\{ \left( -\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \gamma_k \phi_k - \left( \frac{\delta_{zk}}{\pi_k^2} + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} \right) \phi_k^2 \right\} \right. \right. \\ & \quad \left. \left. + \left( \frac{\delta_{zk(l+1)}}{\pi_k^2} + \frac{\delta_{z(k+1)l}}{\pi_{k+1}^2} \right) \phi_k \phi_l \right] \left( \frac{\delta_{zm}}{\pi_m} - \frac{\delta_{z(m+1)}}{\pi_{m+1}} \right) \phi_m \left( \frac{\delta_{zn}}{\pi_n} - \frac{\delta_{z(n+1)}}{\pi_{n+1}} \right) \phi_n \right\} \\ &= \mathbb{E}_X \left[ \delta_{klmn} \left\{ - \left( \gamma_k \phi_k + \frac{\phi_k^2}{\pi_k} \right) \frac{\phi_k^2}{\pi_k^2} + \left( \gamma_k \phi_k - \frac{\phi_k^2}{\pi_{k+1}} \right) \frac{\phi_k^2}{\pi_{k+1}^2} \right\} \right. \\ & \quad \left. + \sum_{(mn)}^2 \left\{ \delta_{klm(n+1)} \left( \gamma_k \phi_k + \frac{\phi_k^2}{\pi_k} \right) \frac{1}{\pi_k^2} \phi_k \phi_{k-1} \right. \right. \\ & \quad \left. \left. - \delta_{(k+1)(l+1)m(n+1)} \left( \gamma_k \phi_k - \frac{\phi_k^2}{\pi_{k+1}} \right) \frac{1}{\pi_{k+1}^2} \phi_{k+1} \phi_k \right\} \right. \\ & \quad \left. + \delta_{kl(m+1)(n+1)} \left\{ - \left( \gamma_k \phi_k + \frac{\phi_k^2}{\pi_k} \right) \frac{\phi_{k-1}^2}{\pi_k^2} \right\} + \delta_{(k+1)(l+1)mn} \left( \gamma_k \phi_k - \frac{\phi_k^2}{\pi_{k+1}} \right) \frac{\phi_{k+1}^2}{\pi_{k+1}^2} \right. \\ & \quad \left. + \sum_{(kl)}^2 \left( \delta_{k(l+1)mn} \frac{1}{\pi_k^3} \phi_k^3 \phi_{k-1} - \sum_{(mn)}^2 \delta_{k(l+1)m(n+1)} \frac{1}{\pi_k^3} \phi_k^2 \phi_{k-1}^2 \right. \right. \\ & \quad \left. \left. + \delta_{k(l+1)(m+1)(n+1)} \frac{1}{\pi_k^3} \phi_k \phi_{k-1}^3 \right) \right], \end{aligned}$$

$$\begin{aligned}
& \mathbf{E} \left( \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \beta_k \partial \beta_l} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_m} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi} \right) \\
&= \mathbf{E} \left\{ \mathbf{x} \left[ \delta_{kl} \left\{ \left( -\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \gamma_k \phi_k - \left( \frac{\delta_{zk}}{\pi_k^2} + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} \right) \phi_k^2 \right\} \right. \right. \\
&\quad \left. \left. + \left( \frac{\delta_{zk(l+1)}}{\pi_k^2} + \frac{\delta_{z(k+1)l}}{\pi_{k+1}^2} \right) \phi_k \phi_l \right] \left( \frac{\delta_{zm}}{\pi_m} - \frac{\delta_{z(m+1)}}{\pi_{m+1}} \right) \phi_m \sum_{a=1}^K \frac{\delta_{za}}{\pi_a} (\phi_a - \phi_{a-1}) \right\} \\
&= \mathbf{E}_X \left( \mathbf{x} \left[ \delta_{klm} \left\{ - \left( \gamma_k \phi_k + \frac{\phi_k^2}{\pi_k} \right) \frac{\phi_k}{\pi_k^2} (\phi_k - \phi_{k-1}) \right. \right. \right. \\
&\quad \left. \left. - \left( \gamma_k \phi_k - \frac{\phi_k^2}{\pi_{k+1}} \right) \frac{\phi_k}{\pi_{k+1}^2} (\phi_{k+1} - \phi_k) \right\} + \delta_{k(l+m+1)} \left( \gamma_k \phi_k + \frac{\phi_k^2}{\pi_k} \right) \frac{\phi_{k-1}}{\pi_k^2} (\phi_k - \phi_{k-1}) \right. \right. \\
&\quad \left. \left. + \delta_{(k+1)(l+1)m} \left( \gamma_k \phi_k - \frac{\phi_k^2}{\pi_{k+1}} \right) \frac{\phi_{k+1}}{\pi_{k+1}^2} (\phi_{k+1} - \phi_k) \right. \right. \\
&\quad \left. \left. + \sum_{(kl)}^2 \left\{ \delta_{k(l+1)m} \frac{1}{\pi_k^3} \phi_k^2 \phi_{k-1} (\phi_k - \phi_{k-1}) - \delta_{k(l+1)(m+1)} \frac{1}{\pi_k^3} \phi_k \phi_{k-1}^2 (\phi_k - \phi_{k-1}) \right\} \right] \right),
\end{aligned}$$

$$\begin{aligned}
& \mathbf{E} \left( \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \beta_k \partial \beta_l} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi'} \right) \\
&= \mathbf{E} \left( \mathbf{x} \mathbf{x}' \left[ \delta_{kl} \left\{ \left( -\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \gamma_k \phi_k - \left( \frac{\delta_{zk}}{\pi_k^2} + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} \right) \phi_k^2 \right\} \right. \right. \\
&\quad \left. \left. + \left( \frac{\delta_{zk(l+1)}}{\pi_k^2} + \frac{\delta_{z(k+1)l}}{\pi_{k+1}^2} \right) \phi_k \phi_l \right] \left\{ \sum_{a=1}^K \frac{\delta_{za}}{\pi_a} (\phi_a - \phi_{a-1}) \right\}^2 \right)
\end{aligned}$$

$$\begin{aligned}
&= \mathbf{E}_X \left( \mathbf{x} \mathbf{x}' \left[ \delta_{kl} \left\{ - \left( \gamma_k \phi_k + \frac{\phi_k^2}{\pi_k} \right) \frac{1}{\pi_k^2} (\phi_k - \phi_{k-1})^2 \right. \right. \right. \\
&+ \left. \left. \left( \gamma_k \phi_k - \frac{\phi_k^2}{\pi_{k+1}} \right) \frac{1}{\pi_{k+1}^2} (\phi_{k+1} - \phi_k)^2 \right\} + \sum_{(kl)}^2 \delta_{k(l+1)} \frac{1}{\pi_k^3} \phi_k \phi_{k-1} (\phi_k - \phi_{k-1})^2 \right] \right), \\
&\mathbf{E} \left( \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \xi \partial \beta_k} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_l} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_m} \right) \\
&= \mathbf{E} \left[ \mathbf{x} \left\{ \left( -\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \gamma_k \phi_k - \frac{\delta_{zk}}{\pi_k^2} \phi_k (\phi_k - \phi_{k-1}) + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} \phi_k (\phi_{k+1} - \phi_k) \right\} \right. \\
&\quad \left. \times \left( \frac{\delta_{zl}}{\pi_l} - \frac{\delta_{z(l+1)}}{\pi_{l+1}} \right) \phi_l \left( \frac{\delta_{zm}}{\pi_m} - \frac{\delta_{z(m+1)}}{\pi_{m+1}} \right) \phi_m \right] \\
&= \mathbf{E}_X \left( \mathbf{x} \left[ \delta_{klm} \left\{ - \left( \gamma_k \phi_k + \frac{1}{\pi_k} \phi_k (\phi_k - \phi_{k-1}) \right) \frac{\phi_k^2}{\pi_k^2} \right. \right. \right. \\
&\quad \left. \left. + \left( \gamma_k \phi_k + \frac{1}{\pi_{k+1}} \phi_k (\phi_{k+1} - \phi_k) \right) \frac{\phi_k^2}{\pi_{k+1}^2} \right\} \right. \\
&+ \sum_{(lm)}^2 \left[ \delta_{kl(m+1)} \left\{ \gamma_k \phi_k + \frac{1}{\pi_k} \phi_k (\phi_k - \phi_{k-1}) \right\} \frac{1}{\pi_k^2} \phi_k \phi_{k-1} \right. \\
&\quad \left. - \delta_{(k+1)l(m+1)} \left\{ \gamma_k \phi_k + \frac{1}{\pi_{k+1}} \phi_k (\phi_{k+1} - \phi_k) \right\} \frac{1}{\pi_{k+1}^2} \phi_{k+1} \phi_k \right] \\
&\quad \left. - \delta_{k(l+1)(m+1)} \left\{ \gamma_k \phi_k + \frac{1}{\pi_k} \phi_k (\phi_k - \phi_{k-1}) \right\} \frac{\phi_{k-1}^2}{\pi_k^2} \right] \right),
\end{aligned}$$

$$\begin{aligned}
& \mathbb{E} \left( \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \xi \partial \beta_k} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_l} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi'} \right) \\
&= \mathbb{E} \left[ \mathbf{x} \mathbf{x}' \left\{ \left( -\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \gamma_k \phi_k - \frac{\delta_{zk}}{\pi_k^2} \phi_k (\phi_k - \phi_{k-1}) + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} \phi_k (\phi_{k+1} - \phi_k) \right\} \right. \\
&\quad \left. \times \left( \frac{\delta_{zl}}{\pi_l} - \frac{\delta_{z(l+1)}}{\pi_{l+1}} \right) \phi_l \sum_{a=1}^K \frac{\delta_{za}}{\pi_a} (\phi_a - \phi_{a-1}) \right] \\
&= \mathbb{E}_X \left( \mathbf{x} \mathbf{x}' \left[ \delta_{kl} \left\{ - \left( \gamma_k \phi_k + \frac{1}{\pi_k} \phi_k (\phi_k - \phi_{k-1}) \right) \frac{\phi_k}{\pi_k^2} (\phi_k - \phi_{k-1}) \right. \right. \right. \\
&\quad \left. \left. - \left( \gamma_k \phi_k + \frac{1}{\pi_{k+1}} \phi_k (\phi_{k+1} - \phi_k) \right) \frac{\phi_k}{\pi_{k+1}^2} (\phi_{k+1} - \phi_k) \right\} \right. \\
&\quad \left. + \delta_{k(l+1)} \left( \gamma_k \phi_k + \frac{1}{\pi_k} \phi_k (\phi_k - \phi_{k-1}) \right) \frac{\phi_{k-1}}{\pi_k^2} (\phi_k - \phi_{k-1}) \right. \\
&\quad \left. \left. + \delta_{(k+1)l} \left( \gamma_k \phi_k + \frac{1}{\pi_{k+1}} \phi_k (\phi_{k+1} - \phi_k) \right) \frac{\phi_{k+1}}{\pi_{k+1}^2} (\phi_{k+1} - \phi_k) \right] \right),
\end{aligned}$$

$$\begin{aligned}
& \mathbb{E} \left( \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \xi \partial \beta_k} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi_l} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi_m} \right) \\
&= \mathbb{E} \left[ \mathbf{x} x_l x_m \left\{ \left( -\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \gamma_k \phi_k - \frac{\delta_{zk}}{\pi_k^2} \phi_k (\phi_k - \phi_{k-1}) + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} \phi_k (\phi_{k+1} - \phi_k) \right\} \right. \\
&\quad \left. \times \left\{ \sum_{a=1}^K \frac{\delta_{za}}{\pi_a} (\phi_a - \phi_{a-1}) \right\}^2 \right]
\end{aligned}$$

$$\begin{aligned}
&= \mathbf{E}_X \left( \mathbf{x} x_l x_m \left[ - \left\{ \gamma_k \phi_k + \frac{1}{\pi_k} \phi_k (\phi_k - \phi_{k-1}) \right\} \frac{1}{\pi_k^2} (\phi_k - \phi_{k-1})^2 \right. \right. \\
&\quad \left. \left. + \left\{ \gamma_k \phi_k + \frac{1}{\pi_{k+1}} \phi_k (\phi_{k+1} - \phi_k) \right\} \frac{1}{\pi_{k+1}^2} (\phi_{k+1} - \phi_k)^2 \right] \right), \\
&\mathbf{E} \left( \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \xi \partial \xi'} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_k} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_l} \right) \\
&= \mathbf{E} \left[ \mathbf{x} \mathbf{x}' \sum_{a=1}^K \left\{ \frac{1}{\pi_a} (-\gamma_a \phi_a + \gamma_{a-1} \phi_{a-1}) - \frac{1}{\pi_a^2} (\phi_a - \phi_{a-1})^2 \right\} \delta_{za} \right. \\
&\quad \left. \times \left( \frac{\delta_{zk}}{\pi_k} - \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \phi_k \left( \frac{\delta_{zl}}{\pi_l} - \frac{\delta_{z(l+1)}}{\pi_{l+1}} \right) \phi_l \right] \\
&= \mathbf{E}_X \left\{ \mathbf{x} \mathbf{x}' \left( \delta_{kl} \left[ \left\{ \frac{1}{\pi_k} (-\gamma_k \phi_k + \gamma_{k-1} \phi_{k-1}) - \frac{1}{\pi_k^2} (\phi_k - \phi_{k-1})^2 \right\} \frac{\phi_k^2}{\pi_k} \right. \right. \right. \\
&\quad \left. \left. + \left\{ \frac{1}{\pi_{k+1}} (-\gamma_{k+1} \phi_{k+1} + \gamma_k \phi_k) - \frac{1}{\pi_{k+1}^2} (\phi_{k+1} - \phi_k)^2 \right\} \frac{\phi_k^2}{\pi_{k+1}} \right] \right. \\
&\quad \left. - \sum_{(kl)}^2 \delta_{k(l+1)} \left\{ \frac{1}{\pi_k} (-\gamma_k \phi_k + \gamma_{k-1} \phi_{k-1}) - \frac{1}{\pi_k^2} (\phi_k - \phi_{k-1})^2 \right\} \frac{\phi_k \phi_{k-1}}{\pi_k} \right) \right\}, \\
&\mathbf{E} \left( \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \xi \partial \xi'} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_k} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi_l} \right) \\
&= \mathbf{E} \left[ \mathbf{x} \mathbf{x}' x_l \sum_{a=1}^K \left\{ \frac{1}{\pi_a} (-\gamma_a \phi_a + \gamma_{a-1} \phi_{a-1}) - \frac{1}{\pi_a^2} (\phi_a - \phi_{a-1})^2 \right\} \delta_{za} \right. \\
&\quad \left. \times \left( \frac{\delta_{zk}}{\pi_k} - \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \phi_k \sum_{b=1}^K \frac{\delta_{zb}}{\pi_b} (\phi_b - \phi_{b-1}) \right]
\end{aligned}$$

$$\begin{aligned}
&= \mathbf{E}_X \left( \mathbf{x} \mathbf{x}' x_l \left[ \left\{ \frac{1}{\pi_k} (-\gamma_k \phi_k + \gamma_{k-1} \phi_{k-1}) - \frac{1}{\pi_k^2} (\phi_k - \phi_{k-1})^2 \right\} \frac{\phi_k}{\pi_k} (\phi_k - \phi_{k-1}) \right. \right. \\
&\quad \left. \left. - \left\{ \frac{1}{\pi_{k+1}} (-\gamma_{k+1} \phi_{k+1} + \gamma_k \phi_k) - \frac{1}{\pi_{k+1}^2} (\phi_{k+1} - \phi_k)^2 \right\} \frac{\phi_k}{\pi_{k+1}} (\phi_{k+1} - \phi_k) \right] \right), \\
&\mathbf{E} \left( \frac{\partial^2 \ln \Pr(Z | \mathbf{x})}{\partial \xi_k \partial \xi_l} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi_m} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi_n} \right) \\
&= \mathbf{E} \left[ x_k x_l x_m x_n \sum_{a=1}^K \left\{ \frac{1}{\pi_a} (-\gamma_a \phi_a + \gamma_{a-1} \phi_{a-1}) - \frac{1}{\pi_a^2} (\phi_a - \phi_{a-1})^2 \right\} \delta_{za} \right. \\
&\quad \left. \times \left\{ \sum_{b=1}^K \frac{\delta_{zb}}{\pi_b} (\phi_b - \phi_{b-1}) \right\}^2 \right] \\
&= \mathbf{E}_X \left[ x_k x_l x_m x_n \sum_{a=1}^K \left\{ (-\gamma_a \phi_a + \gamma_{a-1} \phi_{a-1}) - \frac{1}{\pi_a} (\phi_a - \phi_{a-1})^2 \right\} \frac{1}{\pi_a^2} (\phi_a - \phi_{a-1})^2 \right].
\end{aligned}$$

(2-6)  $\mathbf{E}(N^2 \mathbf{1}_0^{(1)} \mathbf{1}_0^{(3)})$  (for  $\alpha_{\Delta 2}$ )

For  $\mathbf{E}(N^2 \mathbf{1}_0^{(1)<4>})$ ,

$$\mathbf{E} \left[ N^2 \left\{ \frac{\partial^2 \bar{l}}{\partial (\boldsymbol{\theta}_z)_A \partial (\boldsymbol{\theta}_z)_B} - \mathbf{E}(\cdot) \right\} \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_z)_C} \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_z)_D} \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_z)_E} \right],$$

$$\text{and } \mathbf{E} \left[ N^2 \left\{ \frac{\partial^2 \bar{l}}{\partial (\boldsymbol{\theta}_z)_A \partial (\boldsymbol{\theta}_z)_B} - \mathbf{E}(\cdot) \right\} \left\{ \frac{\partial^2 \bar{l}}{\partial (\boldsymbol{\theta}_z)_C \partial (\boldsymbol{\theta}_z)_D} - \mathbf{E}(\cdot) \right\} \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_z)_E} \frac{\partial \bar{l}}{\partial (\boldsymbol{\theta}_z)_F} \right]$$

see (2-3) and (2.4).

The remaining results are

$$\begin{aligned}
&\mathbf{E} \left[ N \left\{ \frac{\partial^3 \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_A \partial (\boldsymbol{\theta}_z)_B \partial (\boldsymbol{\theta}_z)_C} - \mathbf{E}(\cdot) \right\} \frac{\partial \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_D} \right] \\
&= \mathbf{E} \left\{ \frac{\partial^3 \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_A \partial (\boldsymbol{\theta}_z)_B \partial (\boldsymbol{\theta}_z)_C} \frac{\partial \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_D} \right\}
\end{aligned}$$

$$\text{in } \mathbf{E} \left[ N \left\{ \frac{\partial^3 \bar{l}}{\partial(\boldsymbol{\theta}_z)_A \partial(\boldsymbol{\theta}_z)_B \partial(\boldsymbol{\theta}_z)_C} - \mathbf{E}(\cdot) \right\} \frac{\partial \bar{l}}{\partial(\boldsymbol{\theta}_z)_D} \frac{\partial \bar{l}}{\partial(\boldsymbol{\theta}_z)_E} \frac{\partial \bar{l}}{\partial(\boldsymbol{\theta}_z)_F} \right].$$

First, we derive  $\frac{\partial^3 \Pr(Z | \mathbf{x})}{\partial(\boldsymbol{\theta}_z)_A \partial(\boldsymbol{\theta}_z)_B \partial(\boldsymbol{\theta}_z)_C}$  as follows:

$$\begin{aligned} \frac{\partial^3 \ln \Pr(Z | \mathbf{x})}{\partial \beta_k \partial \beta_l \partial \beta_m} &= \delta_{klm} \left\{ \left( -\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) (1 - \gamma_k^2) \phi_k \right. \\ &\quad \left. + 3 \left( \frac{\delta_{zk}}{\pi_k^2} + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} \right) \gamma_k \phi_k^2 + 2 \left( \frac{\delta_{zk}}{\pi_k^3} - \frac{\delta_{z(k+1)}}{\pi_{k+1}^3} \right) \phi_k^3 \right\} \\ &+ \sum_{(klm)}^3 \left\{ \delta_{z(k+1)lm} \left( -\frac{1}{\pi_{k+1}^2} \gamma_{k+1} \phi_{k+1} \phi_k - \frac{2}{\pi_{k+1}^3} \phi_{k+1}^2 \phi_k \right) \right. \\ &\quad \left. + \delta_{z(k+1)(l+1)m} \left( -\frac{1}{\pi_{k+1}^2} \gamma_k \phi_k \phi_{k+1} + \frac{2}{\pi_{k+1}^3} \phi_k^2 \phi_{k+1} \right) \right\}, \\ \frac{\partial^3 \ln \Pr(Z | \mathbf{x})}{\partial \beta_k \partial \beta_l \partial \xi} &= \mathbf{x} \left[ \delta_{kl} \left\{ \left( -\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) (1 - \gamma_k^2) \phi_k \right. \right. \\ &\quad \left. \left. + \left( \frac{\delta_{zk}}{\pi_k^2} (\phi_k - \phi_{k-1}) - \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} (\phi_{k+1} - \phi_k) \right) \gamma_k \phi_k \right. \right. \\ &\quad \left. \left. + 2 \left( \frac{\delta_{zk}}{\pi_k^2} + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} \right) \gamma_k \phi_k^2 + 2 \left( \frac{\delta_{zk}}{\pi_k^3} (\phi_k - \phi_{k-1}) + \frac{\delta_{z(k+1)}}{\pi_{k+1}^3} (\phi_{k+1} - \phi_k) \right) \phi_k^2 \right\} \right. \\ &\quad \left. - \left( \frac{\delta_{zk(l+1)}}{\pi_k^2} + \frac{\delta_{z(k+1)l}}{\pi_{k+1}^2} \right) (\gamma_k + \gamma_l) \phi_k \phi_l \right. \\ &\quad \left. - 2 \left( \frac{\delta_{zk(l+1)}}{\pi_k^3} (\phi_k - \phi_{k-1}) + \frac{\delta_{z(k+1)l}}{\pi_{k+1}^3} (\phi_{k+1} - \phi_k) \right) \phi_k \phi_l \right], \end{aligned}$$

$$\begin{aligned}
\frac{\partial^3 \ln \Pr(Z | \mathbf{x})}{\partial \beta_k \partial \xi \partial \xi'} &= \mathbf{x} \mathbf{x}' \left[ \left( -\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) (1 - \gamma_k^2) \phi_k \right. \\
&\quad \left. + \left\{ \frac{\delta_{zk}}{\pi_k^2} (\phi_k - \phi_{k-1}) - \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} (\phi_{k+1} - \phi_k) \right\} \gamma_k \phi_k \right. \\
&\quad \left. + \frac{\delta_{zk}}{\pi_k^2} \{ 2\gamma_k \phi_k^2 - (\gamma_k + \gamma_{k-1}) \phi_k \phi_{k-1} \} + \frac{\delta_{zk}}{\pi_k^3} 2\phi_k (\phi_k - \phi_{k-1})^2 \right. \\
&\quad \left. + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} \{ 2\gamma_k \phi_k^2 - (\gamma_{k+1} + \gamma_k) \phi_{k+1} \phi_k \} - \frac{\delta_{z(k+1)}}{\pi_{k+1}^3} 2\phi_k (\phi_{k+1} - \phi_k)^2 \right], \\
\frac{\partial^3 \ln \Pr(Z | \mathbf{x})}{\partial \xi_k \partial \xi_l \partial \xi_m} &= x_k x_l x_m \sum_{a=1}^K \left[ \frac{1}{\pi_a} \{ (-1 + \gamma_a^2) \phi_a + (1 - \gamma_{a-1}^2) \phi_{a-1} \} \right. \\
&\quad \left. + \frac{3}{\pi_a^2} (\gamma_a \phi_a - \gamma_{a-1} \phi_{a-1}) (\phi_a - \phi_{a-1}) + \frac{2}{\pi_a^3} (\phi_a - \phi_{a-1})^3 \right] \delta_{za}.
\end{aligned}$$

Then, the expectations are

$$\begin{aligned}
\mathbb{E} \left( \frac{\partial^3 \ln \Pr(Z | \mathbf{x})}{\partial \beta_k \partial \beta_l \partial \beta_m} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_n} \right) &= \mathbb{E} \left( \left[ \delta_{klm} \left\{ \left( -\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) (1 - \gamma_k^2) \phi_k \right. \right. \right. \\
&\quad \left. \left. + 3 \left( \frac{\delta_{zk}}{\pi_k^2} + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} \right) \gamma_k \phi_k^2 + 2 \left( \frac{\delta_{zk}}{\pi_k^3} - \frac{\delta_{z(k+1)}}{\pi_{k+1}^3} \right) \phi_k^3 \right\} \right. \\
&\quad \left. + \sum_{(klm)}^3 \left\{ \delta_{z(k+1)lm} \left( -\frac{1}{\pi_{k+1}^2} \gamma_{k+1} \phi_{k+1} \phi_k - \frac{2}{\pi_{k+1}^3} \phi_{k+1}^2 \phi_k \right) \right. \right. \\
&\quad \left. \left. + \delta_{z(k+1)(l+1)m} \left( -\frac{1}{\pi_{k+1}^2} \gamma_k \phi_k \phi_{k+1} + \frac{2}{\pi_{k+1}^3} \phi_k^2 \phi_{k+1} \right) \right\} \right] \left( \frac{\delta_{zn}}{\pi_n} - \frac{\delta_{z(n+1)}}{\pi_{n+1}} \right) \phi_n \right)
\end{aligned}$$

$$\begin{aligned}
&= \mathbf{E}_X \left[ \delta_{klmn} \left\{ - \left( \frac{1}{\pi_k} + \frac{1}{\pi_{k+1}} \right) (1 - \gamma_k^2) \phi_k^2 + 3 \left( \frac{1}{\pi_k^2} - \frac{1}{\pi_{k+1}^2} \right) \gamma_k \phi_k^3 \right. \right. \\
&\quad \left. \left. + 2 \left( \frac{1}{\pi_k^3} + \frac{1}{\pi_{k+1}^3} \right) \phi_k^4 \right\} \right. \\
&+ \sum_{(klm)}^3 \left\{ \delta_{(k+1)lmn} \left( - \frac{1}{\pi_{k+1}^2} \gamma_{k+1} \phi_{k+1}^2 \phi_k - \frac{2}{\pi_{k+1}^3} \phi_{k+1}^3 \phi_k \right) \right. \\
&\quad + \delta_{(k+1)lm(n+1)} \left( \frac{1}{\pi_{k+1}^2} \gamma_{k+1} \phi_{k+1} \phi_k^2 + \frac{2}{\pi_{k+1}^3} \phi_{k+1}^2 \phi_k^2 \right) \\
&\quad + \delta_{(k+1)(l+1)mn} \left( - \frac{1}{\pi_{k+1}^2} \gamma_k \phi_{k+1}^2 \phi_k + \frac{2}{\pi_{k+1}^3} \phi_{k+1}^2 \phi_k^2 \right) \\
&\quad \left. + \delta_{(k+1)(l+1)m(n+1)} \left( \frac{1}{\pi_{k+1}^2} \gamma_k \phi_{k+1} \phi_k^2 - \frac{2}{\pi_{k+1}^3} \phi_{k+1} \phi_k^3 \right) \right\} \\
&\quad + \delta_{klm(n+1)} \left\{ \frac{1}{\pi_k} (1 - \gamma_k^2) \phi_k - \frac{3}{\pi_k^2} \gamma_k \phi_k^2 - \frac{2}{\pi_k^3} \phi_k^3 \right\} \phi_{k-1} \\
&\quad \left. + \delta_{(k+1)(l+1)(m+1)n} \left\{ \frac{1}{\pi_{k+1}} (1 - \gamma_k^2) \phi_k + \frac{3}{\pi_{k+1}^2} \gamma_k \phi_k^2 - \frac{2}{\pi_{k+1}^3} \phi_k^3 \right\} \phi_{k+1} \right],
\end{aligned}$$

$$\begin{aligned}
\mathbf{E} \left( \frac{\partial^3 \ln \Pr(Z | \mathbf{x})}{\partial \beta_k \partial \beta_l \partial \beta_m} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi} \right) &= \mathbf{E} \left\{ \mathbf{x} \left[ \delta_{klm} \left\{ \left( - \frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) (1 - \gamma_k^2) \phi_k \right. \right. \right. \\
&\quad \left. \left. + 3 \left( \frac{\delta_{zk}}{\pi_k^2} + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} \right) \gamma_k \phi_k^2 + 2 \left( \frac{\delta_{zk}}{\pi_k^3} - \frac{\delta_{z(k+1)}}{\pi_{k+1}^3} \right) \phi_k^3 \right\} \right.
\end{aligned}$$

$$\begin{aligned}
& + \sum_{(klm)}^3 \left\{ \delta_{z(k+1)lm} \left( -\frac{1}{\pi_{k+1}^2} \gamma_{k+1} \phi_{k+1} \phi_k - \frac{2}{\pi_{k+1}^3} \phi_{k+1}^2 \phi_k \right) \right. \\
& + \left. \delta_{z(k+1)(l+1)m} \left( -\frac{1}{\pi_{k+1}^2} \gamma_k \phi_k \phi_{k+1} + \frac{2}{\pi_{k+1}^3} \phi_k^2 \phi_{k+1} \right) \right\} \left[ \sum_{a=1}^K \frac{\delta_{za}}{\pi_a} (\phi_a - \phi_{a-1}) \right] \\
& = \mathbf{E}_{\mathbf{X}} \left\{ \mathbf{x} \left( \delta_{klm} \left[ \left\{ -\frac{1}{\pi_k} (1 - \gamma_k^2) \phi_k + \frac{3}{\pi_k^2} \gamma_k \phi_k^2 + \frac{2}{\pi_k^3} \phi_k^3 \right\} (\phi_k - \phi_{k-1}) \right. \right. \right. \\
& \quad \left. \left. \left. + \left\{ \frac{1}{\pi_{k+1}} (1 - \gamma_k^2) \phi_k + \frac{3}{\pi_{k+1}^2} \gamma_k \phi_k^2 - \frac{2}{\pi_{k+1}^3} \phi_k^3 \right\} (\phi_{k+1} - \phi_k) \right] \right) \right. \\
& \quad \left. + \sum_{(klm)}^3 \left\{ \delta_{(k+1)lm} \left( -\frac{1}{\pi_{k+1}^2} \gamma_{k+1} \phi_{k+1} \phi_k - \frac{2}{\pi_{k+1}^3} \phi_{k+1}^2 \phi_k \right) \right. \right. \\
& \quad \left. \left. + \delta_{(k+1)(l+1)m} \left( -\frac{1}{\pi_{k+1}^2} \gamma_k \phi_k \phi_{k+1} + \frac{2}{\pi_{k+1}^3} \phi_k^2 \phi_{k+1} \right) \right\} (\phi_{k+1} - \phi_k) \right\},
\end{aligned}$$

$$\begin{aligned}
\mathbf{E} \left( \frac{\partial^3 \ln \Pr(Z | \mathbf{x})}{\partial \beta_k \partial \beta_l \partial \xi} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_m} \right) & = \mathbf{E} \left\{ \mathbf{x} \left[ \delta_{kl} \left\{ \left( -\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) (1 - \gamma_k^2) \phi_k \right. \right. \right. \\
& + \left. \left( \frac{\delta_{zk}}{\pi_k^2} (\phi_k - \phi_{k-1}) - \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} (\phi_{k+1} - \phi_k) \right) \gamma_k \phi_k + 2 \left( \frac{\delta_{zk}}{\pi_k^2} + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} \right) \gamma_k \phi_k^2 \right. \right. \\
& \left. \left. + 2 \left( \frac{\delta_{zk}}{\pi_k^3} (\phi_k - \phi_{k-1}) + \frac{\delta_{z(k+1)}}{\pi_{k+1}^3} (\phi_{k+1} - \phi_k) \right) \phi_k^2 \right\} \right. \\
& \left. - \sum_{(kl)}^2 \delta_{zk(l+1)} \left\{ \frac{1}{\pi_k} (\gamma_k + \gamma_{k-1}) + \frac{2}{\pi_k^3} (\phi_k - \phi_{k-1}) \right\} \phi_k \phi_{k-1} \right] \left( \frac{\delta_{zm}}{\pi_m} - \frac{\delta_{z(m+1)}}{\pi_{m+1}} \right) \phi_m \right\}
\end{aligned}$$

$$\begin{aligned}
&= \mathbf{E}_X \left\{ \mathbf{x} \left( \delta_{klm} \left\{ - \left( \frac{1}{\pi_k} + \frac{1}{\pi_{k+1}} \right) (1 - \gamma_k^2) \phi_k^2 \right. \right. \right. \\
&\quad \left. \left. + \left( \frac{1}{\pi_k^2} (\phi_k - \phi_{k-1}) + \frac{1}{\pi_{k+1}^2} (\phi_{k+1} - \phi_k) \right) \gamma_k \phi_k^2 + 2 \left( \frac{1}{\pi_k^2} - \frac{1}{\pi_{k+1}^2} \right) \gamma_k \phi_k^3 \right. \right. \\
&\quad \left. \left. + 2 \left( \frac{1}{\pi_k^3} (\phi_k - \phi_{k-1}) - \frac{1}{\pi_{k+1}^3} (\phi_{k+1} - \phi_k) \right) \phi_k^3 \right\} \right. \\
&\quad \left. + \sum_{(kl)}^2 \left[ -\delta_{k(l+1)m} \left\{ \frac{1}{\pi_k^2} (\gamma_k + \gamma_{k-1}) + \frac{2}{\pi_k^3} (\phi_k - \phi_{k-1}) \right\} \phi_k^2 \phi_{k-1} \right. \right. \\
&\quad \left. \left. + \delta_{k(l+1)(m+1)} \left\{ \frac{1}{\pi_k^2} (\gamma_k + \gamma_{k-1}) + \frac{2}{\pi_k^3} (\phi_k - \phi_{k-1}) \right\} \phi_k \phi_{k-1}^2 \right] \right\},
\end{aligned}$$

$$\begin{aligned}
&\mathbf{E} \left( \frac{\partial^3 \ln \Pr(Z | \mathbf{x})}{\partial \beta_k \partial \beta_l \partial \xi} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi'} \right) = \mathbf{E} \left\{ \mathbf{x} \mathbf{x}' \left[ \delta_{kl} \left\{ \left( -\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) (1 - \gamma_k^2) \phi_k \right. \right. \right. \\
&\quad \left. \left. + \left( \frac{\delta_{zk}}{\pi_k^2} (\phi_k - \phi_{k-1}) - \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} (\phi_{k+1} - \phi_k) \right) \gamma_k \phi_k + 2 \left( \frac{\delta_{zk}}{\pi_k^2} + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} \right) \gamma_k \phi_k^2 \right. \right. \\
&\quad \left. \left. + 2 \left( \frac{\delta_{zk}}{\pi_k^3} (\phi_k - \phi_{k-1}) + \frac{\delta_{z(k+1)}}{\pi_{k+1}^3} (\phi_{k+1} - \phi_k) \right) \phi_k^2 \right\} \right. \\
&\quad \left. - \sum_{(kl)}^2 \delta_{zk(l+1)} \left\{ \frac{1}{\pi_k^2} (\gamma_k + \gamma_{k-1}) + \frac{2}{\pi_k^3} (\phi_k - \phi_{k-1}) \right\} \phi_k \phi_{k-1} \right] \sum_{a=1}^K \frac{\delta_{za}}{\pi_a} (\phi_a - \phi_{a-1}) \left. \right\} \\
&= \mathbf{E}_X \left( \mathbf{x} \mathbf{x}' \left\{ \delta_{kl} \left[ \left\{ -\frac{1}{\pi_k} (1 - \gamma_k^2) \phi_k + \frac{1}{\pi_k^2} (\phi_k - \phi_{k-1}) \gamma_k \phi_k + \frac{2}{\pi_k^2} \gamma_k \phi_k^2 \right. \right. \right. \right. \\
&\quad \left. \left. + \frac{2}{\pi_k^3} (\phi_k - \phi_{k-1}) \phi_k^2 \right\} (\phi_k - \phi_{k-1}) + \left\{ \frac{1}{\pi_{k+1}} (1 - \gamma_k^2) \phi_k - \frac{1}{\pi_{k+1}^2} (\phi_{k+1} - \phi_k) \gamma_k \phi_k \right. \right. \right. \\
&\quad \left. \left. + \frac{2}{\pi_{k+1}^2} \gamma_k \phi_k^2 \right\} (\phi_{k+1} - \phi_k) \right. \left. \left. \right\} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{2}{\pi_{k+1}^2} \gamma_k \phi_k^2 + \frac{2}{\pi_{k+1}^3} (\phi_{k+1} - \phi_k) \phi_k^2 \left. \vphantom{\frac{2}{\pi_{k+1}^2}} \right\} (\phi_{k+1} - \phi_k) \left. \vphantom{\frac{2}{\pi_{k+1}^2}} \right] \\
& - \sum_{(kl)}^2 \delta_{k(l+1)} \left\{ \frac{1}{\pi_k^2} (\gamma_k + \gamma_{k-1}) + \frac{2}{\pi_k^3} (\phi_k - \phi_{k-1}) \right\} \phi_k \phi_{k-1} (\phi_k - \phi_{k-1}) \left. \vphantom{\frac{1}{\pi_k^2}} \right\},
\end{aligned}$$

$$\begin{aligned}
& \mathbf{E} \left( \frac{\partial^3 \ln \Pr(Z | \mathbf{x})}{\partial \beta_k \partial \xi \partial \xi'} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_l} \right) = \mathbf{E} \left\{ \mathbf{x} \mathbf{x}' \left[ \left( -\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) (1 - \gamma_k^2) \phi_k \right. \right. \\
& + \left. \left. \left\{ \frac{\delta_{zk}}{\pi_k^2} (\phi_k - \phi_{k-1}) - \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} (\phi_{k+1} - \phi_k) \right\} \gamma_k \phi_k \right. \right. \\
& + \left. \left. \frac{\delta_{zk}}{\pi_k^2} \{ 2\gamma_k \phi_k^2 - (\gamma_k + \gamma_{k+1}) \phi_k \phi_{k-1} \} + \frac{\delta_{zk}}{\pi_k^3} 2\phi_k (\phi_k - \phi_{k-1})^2 \right. \right. \\
& + \left. \left. \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} \{ 2\gamma_k \phi_k^2 - (\gamma_{k+1} + \gamma_k) \phi_{k+1} \phi_k \} - \frac{\delta_{z(k+1)}}{\pi_{k+1}^3} 2\phi_k (\phi_{k+1} - \phi_k)^2 \right] \right. \\
& \left. \times \left( \frac{\delta_{zl}}{\pi_l} - \frac{\delta_{z(l+1)}}{\pi_{l+1}} \right) \phi_l \right\} \\
& = \mathbf{E}_X \left\{ \mathbf{x} \mathbf{x}' \left( \delta_{kl} \left[ - \left( \frac{1}{\pi_k} + \frac{1}{\pi_{k+1}} \right) (1 - \gamma_k^2) \phi_k^2 \right. \right. \right. \\
& + \left. \left. \left\{ \frac{1}{\pi_k^2} (\phi_k - \phi_{k-1}) + \frac{1}{\pi_{k+1}^2} (\phi_{k+1} - \phi_k) \right\} \gamma_k \phi_k^2 \right. \right. \\
& + \left. \left. \frac{1}{\pi_k^2} \{ 2\gamma_k \phi_k^3 - (\gamma_k + \gamma_{k-1}) \phi_k^2 \phi_{k-1} \} + \frac{1}{\pi_k^3} 2\phi_k^2 (\phi_k - \phi_{k-1})^2 \right. \right. \\
& \left. \left. - \frac{1}{\pi_{k+1}^2} \{ 2\gamma_k \phi_k^3 - (\gamma_{k+1} + \gamma_k) \phi_{k+1} \phi_k^2 \} + \frac{1}{\pi_{k+1}^3} 2\phi_k^2 (\phi_{k+1} - \phi_k)^2 \right] \right. \\
& \left. \right\}
\end{aligned}$$

$$\begin{aligned}
& +\delta_{k(l+1)} \left[ \frac{1}{\pi_k} (1-\gamma_k^2) \phi_k \phi_{k-1} - \frac{1}{\pi_k^2} (\phi_k - \phi_{k-1}) \gamma_k \phi_k \phi_{k-1} \right. \\
& \left. - \frac{1}{\pi_k^2} \{2\gamma_k \phi_k^2 - (\gamma_k + \gamma_{k-1}) \phi_k \phi_{k-1}\} \phi_{k-1} - \frac{2}{\pi_k^3} \phi_k \phi_{k-1} (\phi_k - \phi_{k-1})^2 \right] \\
& +\delta_{(k+1)l} \left[ \frac{1}{\pi_{k+1}} (1-\gamma_k^2) \phi_k \phi_{k+1} - \frac{1}{\pi_{k+1}^2} (\phi_{k+1} - \phi_k) \gamma_k \phi_k \phi_{k+1} \right. \\
& \left. + \frac{1}{\pi_{k+1}^2} \{2\gamma_k \phi_k^2 - (\gamma_{k+1} + \gamma_k) \phi_{k+1} \phi_k\} \phi_{k+1} - \frac{2}{\pi_{k+1}^3} \phi_k \phi_{k+1} (\phi_{k+1} - \phi_k)^2 \right] \Bigg\},
\end{aligned}$$

$$\begin{aligned}
& \mathbb{E} \left( \frac{\partial^3 \ln \Pr(Z | \mathbf{x})}{\partial \beta_k \partial \xi \partial \xi'} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi_l} \right) = \mathbb{E} \left\{ \mathbf{x} \mathbf{x}' x_l \left[ \left( -\frac{\delta_{zk}}{\pi_k} + \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) (1-\gamma_k^2) \phi_k \right. \right. \\
& \left. \left. + \left\{ \frac{\delta_{zk}}{\pi_k^2} (\phi_k - \phi_{k-1}) - \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} (\phi_{k+1} - \phi_k) \right\} \gamma_k \phi_k \right. \right. \\
& \left. \left. + \frac{\delta_{zk}}{\pi_k^2} \{2\gamma_k \phi_k^2 - (\gamma_k + \gamma_{k-1}) \phi_k \phi_{k-1}\} + \frac{\delta_{zk}}{\pi_k^3} 2\phi_k (\phi_k - \phi_{k-1})^2 \right. \right. \\
& \left. \left. + \frac{\delta_{z(k+1)}}{\pi_{k+1}^2} \{2\gamma_k \phi_k^2 - (\gamma_{k+1} + \gamma_k) \phi_{k+1} \phi_k\} - \frac{\delta_{z(k+1)}}{\pi_{k+1}^3} 2\phi_k (\phi_{k+1} - \phi_k)^2 \right] \right. \\
& \left. \times \sum_{a=1}^K \frac{\delta_{za}}{\pi_a} (\phi_a - \phi_{a-1}) \right\} \\
& = \mathbb{E}_X \left( \mathbf{x} \mathbf{x}' x_l \left\{ \left[ -\frac{1}{\pi_k} (1-\gamma_k^2) \phi_k + \frac{1}{\pi_k^2} (\phi_k - \phi_{k-1}) \gamma_k \phi_k \right. \right. \right. \\
& \left. \left. + \frac{1}{\pi_k^2} \{2\gamma_k \phi_k^2 - (\gamma_k + \gamma_{k-1}) \phi_k \phi_{k-1}\} + \frac{2}{\pi_k^3} \phi_k (\phi_k - \phi_{k-1})^2 \right] (\phi_k - \phi_{k-1}) \right\} \right)
\end{aligned}$$

$$\begin{aligned}
& + \left[ \frac{1}{\pi_{k+1}} (1 - \gamma_k^2) \phi_k - \frac{1}{\pi_{k+1}^2} (\phi_{k+1} - \phi_k) \gamma_k \phi_k \right. \\
& \left. + \frac{1}{\pi_{k+1}^2} \{ 2\gamma_k \phi_k^2 - (\gamma_{k+1} + \gamma_k) \phi_{k+1} \phi_k \} - \frac{2}{\pi_{k+1}^3} \phi_k (\phi_{k+1} - \phi_k)^2 \right] (\phi_{k+1} - \phi_k) \Bigg\},
\end{aligned}$$

$$\begin{aligned}
& \mathbb{E} \left( \frac{\partial^3 \ln \Pr(Z | \mathbf{x})}{\partial \xi_k \partial \xi_l \partial \xi_m} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_n} \right) \\
& = \mathbb{E} \left\{ x_k x_l x_m \sum_{a=1}^K \left[ \frac{1}{\pi_a} \{ (-1 + \gamma_a^2) \phi_a + (1 - \gamma_{a-1}^2) \phi_{a-1} \} \right. \right. \\
& \left. \left. + \frac{3}{\pi_a^2} (\gamma_a \phi_a - \gamma_{a-1} \phi_{a-1}) (\phi_a - \phi_{a-1}) + \frac{2}{\pi_a^3} (\phi_a - \phi_{a-1})^3 \right] \delta_{za} \left( \frac{\delta_{zn}}{\pi_n} - \frac{\delta_{z(n+1)}}{\pi_{n+1}} \right) \phi_n \right\} \\
& = \mathbb{E}_X \left( x_k x_l x_m \left[ \frac{1}{\pi_n} \{ (-1 + \gamma_n^2) \phi_n + (1 - \gamma_{n-1}^2) \phi_{n-1} \} \phi_n \right. \right. \\
& \left. \left. + \frac{3}{\pi_n^2} (\gamma_n \phi_n - \gamma_{n-1} \phi_{n-1}) (\phi_n - \phi_{n-1}) \phi_n + \frac{2}{\pi_n^3} (\phi_n - \phi_{n-1})^3 \phi_n \right. \right. \\
& \left. \left. - \frac{1}{\pi_{n+1}} \{ (-1 + \gamma_{n+1}^2) \phi_{n+1} + (1 - \gamma_n^2) \phi_n \} \phi_n \right. \right. \\
& \left. \left. - \frac{3}{\pi_{n+1}^2} (\gamma_{n+1} \phi_{n+1} - \gamma_n \phi_n) (\phi_{n+1} - \phi_n) \phi_n - \frac{2}{\pi_{n+1}^3} (\phi_{n+1} - \phi_n)^3 \phi_n \right] \right),
\end{aligned}$$

$$\mathbb{E} \left( \frac{\partial^3 \ln \Pr(Z | \mathbf{x})}{\partial \xi_k \partial \xi_l \partial \xi_m} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi_n} \right)$$

$$\begin{aligned}
&= \mathbf{E} \left\{ x_k x_l x_m x_n \sum_{a=1}^K \left[ \frac{1}{\pi_a} \{(-1 + \gamma_a^2)\phi_a + (1 - \gamma_{a-1}^2)\phi_{a-1}\} \right. \right. \\
&\quad \left. \left. + \frac{3}{\pi_a^2} (\gamma_a \phi_a - \gamma_{a-1} \phi_{a-1})(\phi_a - \phi_{a-1}) + \frac{2}{\pi_a^3} (\phi_a - \phi_{a-1})^3 \right] \delta_{za} \sum_{b=1}^K \frac{\delta_{zb}}{\pi_b} (\phi_b - \phi_{b-1}) \right\} \\
&= \mathbf{E}_X \left\{ x_k x_l x_m x_n \sum_{a=1}^K \left[ \frac{1}{\pi_a} \{(-1 + \gamma_a^2)\phi_a + (1 - \gamma_{a-1}^2)\phi_{a-1}\} \right. \right. \\
&\quad \left. \left. + \frac{3}{\pi_a^2} (\gamma_a \phi_a - \gamma_{a-1} \phi_{a-1})(\phi_a - \phi_{a-1}) + \frac{2}{\pi_a^3} (\phi_a - \phi_{a-1})^3 \right] (\phi_a - \phi_{a-1}) \right\}.
\end{aligned}$$

$$(2-7) \quad \mathbf{E}(N^3 \mathbf{I}_0^{(1)<4>}) - N \sum^3 \{ \mathbf{E}(N \mathbf{I}_0^{(1)<2>}) \}^{<2>} \quad (\text{for } \alpha_4)$$

$$\begin{aligned}
&\frac{N^2 - N}{N} \sum^3 \mathbf{E} \left\{ \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_A} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_B} \right\} \\
&\quad \times \mathbf{E} \left\{ \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_C} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_D} \right\} + \mathbf{E} \left\{ \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_A} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_B} \right. \\
&\quad \quad \left. \times \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_C} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_D} \right\} - N \sum^3 (\boldsymbol{\Lambda}_z)_{AB} (\boldsymbol{\Lambda}_z)_{CD} \\
&= \mathbf{E} \left\{ \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_A} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_B} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_C} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial (\boldsymbol{\theta}_z)_D} \right\} \\
&\quad - \sum^3 (\boldsymbol{\Lambda}_z)_{AB} (\boldsymbol{\Lambda}_z)_{CD}
\end{aligned}$$

(the fourth multivariate cumulant of  $\partial \ln \Pr(Z | \mathbf{x}) / \partial (\boldsymbol{\theta}_z)_A$ 's).

The expectations required in the above cumulants are

$$\mathbf{E} \left\{ \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_k} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_l} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_m} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_n} \right\}$$

$$\begin{aligned}
&= \mathbf{E} \left\{ \left( \frac{\delta_{zk}}{\pi_k} - \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \phi_k \left( \frac{\delta_{zl}}{\pi_l} - \frac{\delta_{z(l+1)}}{\pi_{l+1}} \right) \phi_l \right. \\
&\quad \left. \times \left( \frac{\delta_{zm}}{\pi_m} - \frac{\delta_{z(m+1)}}{\pi_{m+1}} \right) \phi_m \left( \frac{\delta_{zn}}{\pi_n} - \frac{\delta_{z(n+1)}}{\pi_{n+1}} \right) \phi_n \right\} \\
&= \mathbf{E}_X \left[ \delta_{klmn} \left( \frac{1}{\pi_k^3} + \frac{1}{\pi_{k+1}^3} \right) \phi_k^4 - \sum_{(klmn)}^4 \left\{ \delta_{klm(n+1)} \frac{1}{\pi_k^3} \phi_k^3 \phi_{k-1} \right. \right. \\
&\quad \left. \left. + \delta_{(k+1)(l+1)(m+1)n} \frac{1}{\pi_{k+1}^3} \phi_{k+1} \phi_k^3 \right\} + \sum_{(klmn)}^6 \delta_{kl(m+1)(n+1)} \frac{1}{\pi_k^3} \phi_k^2 \phi_{k-1}^2 \right],
\end{aligned}$$

$$\begin{aligned}
&\mathbf{E} \left\{ \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_k} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_l} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_m} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi} \right\} \\
&= \mathbf{E} \left\{ \mathbf{x} \left( \frac{\delta_{zk}}{\pi_k} - \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \phi_k \left( \frac{\delta_{zl}}{\pi_l} - \frac{\delta_{z(l+1)}}{\pi_{l+1}} \right) \phi_l \right. \\
&\quad \left. \times \left( \frac{\delta_{zm}}{\pi_m} - \frac{\delta_{z(m+1)}}{\pi_{m+1}} \right) \phi_m \sum_{a=1}^K \frac{\delta_{za}}{\pi_a} (\phi_a - \phi_{a-1}) \right\} \\
&= \mathbf{E}_X \left( \mathbf{x} \left[ \delta_{klm} \left\{ \frac{1}{\pi_k^3} \phi_k^3 (\phi_k - \phi_{k-1}) - \frac{1}{\pi_{k+1}^3} \phi_k^3 (\phi_{k+1} - \phi_k) \right\} \right. \right. \\
&\quad \left. \left. + \sum_{(klm)}^3 \left\{ -\delta_{kl(m+1)} \frac{1}{\pi_k^3} \phi_k^2 \phi_{k-1} (\phi_k - \phi_{k-1}) + \delta_{(k+1)(l+1)m} \frac{1}{\pi_{k+1}^3} \phi_k^2 \phi_{k+1} (\phi_{k+1} - \phi_k) \right\} \right] \right),
\end{aligned}$$

$$\begin{aligned}
&\mathbf{E} \left\{ \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_k} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_l} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi'} \right\} \\
&= \mathbf{E} \left[ \mathbf{x} \mathbf{x}' \left( \frac{\delta_{zk}}{\pi_k} - \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \phi_k \left( \frac{\delta_{zl}}{\pi_l} - \frac{\delta_{z(l+1)}}{\pi_{l+1}} \right) \phi_l \left\{ \sum_{a=1}^K \frac{\delta_{za}}{\pi_a} (\phi_a - \phi_{a-1}) \right\}^2 \right]
\end{aligned}$$

$$\begin{aligned}
&= \mathbf{E}_X \left( \mathbf{x} \mathbf{x}' \left[ \delta_{kl} \left\{ \frac{1}{\pi_k^3} (\phi_k - \phi_{k-1})^2 + \frac{1}{\pi_{k+1}^3} (\phi_{k+1} - \phi_k)^2 \right\} \phi_k^2 \right. \right. \\
&\quad \left. \left. - \sum_{(kl)}^2 \delta_{k(l+1)} \frac{1}{\pi_k^3} \phi_k \phi_{k-1} (\phi_k - \phi_{k-1})^2 \right] \right), \\
&\mathbf{E} \left\{ \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \beta_k} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi_l} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi_m} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi_n} \right\} \\
&= \mathbf{E} \left[ x_l x_m x_n \left( \frac{\delta_{zk}}{\pi_k} - \frac{\delta_{z(k+1)}}{\pi_{k+1}} \right) \phi_k \left\{ \sum_{a=1}^K \frac{\delta_{za}}{\pi_a} (\phi_a - \phi_{a-1}) \right\}^3 \right] \\
&= \mathbf{E}_X \left[ x_l x_m x_n \left\{ \frac{1}{\pi_k^3} \phi_k (\phi_k - \phi_{k-1})^3 - \frac{1}{\pi_{k+1}^3} \phi_k (\phi_{k+1} - \phi_k)^3 \right\} \right], \\
&\mathbf{E} \left\{ \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi_k} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi_l} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi_m} \frac{\partial \ln \Pr(Z | \mathbf{x})}{\partial \xi_n} \right\} \\
&= \mathbf{E} \left[ x_k x_l x_m x_n \left\{ \sum_{a=1}^K \frac{\delta_{za}}{\pi_a} (\phi_a - \phi_{a-1}) \right\}^4 \right] \\
&= \mathbf{E}_X \left[ x_k x_l x_m x_n \sum_{a=1}^K \frac{1}{\pi_a^3} (\phi_a - \phi_{a-1})^4 \right],
\end{aligned}$$

$$(2-8) \quad \mathbf{E}(N^3 \mathbf{I}_0^{(1)<3>} \otimes \mathbf{I}_0^{(2)}) \quad (\text{for } \alpha_4)$$

The expectations are expressed as  $\sum^{10} (\cdot)$ , where each  $(\cdot)$  is the product of two expectations (see (2-2), (2-3), and (2-5)).

$$(2-9) \quad \mathbf{E}(N^3 \mathbf{I}_0^{(1)<2>} \otimes \mathbf{I}_0^{(2)<2>}) \quad (\text{for } \alpha_4)$$

The results are expressed as  $\sum^{15}(\cdot)$ , where each  $(\cdot)$  is the product of three expectations (see (2-1) and (2-4) ).

$$(2-10) \quad E(N^3 \mathbf{1}_0^{(1)<3>} \otimes \mathbf{I}_0^{(3)}) \quad (\text{for } \alpha_4)$$

The results are similar to those in (2-9) (see (2-6)).

$$(2-11) \quad E\left(\frac{\partial^3 \bar{l}}{\partial(\boldsymbol{\theta}_z)_A \partial(\boldsymbol{\theta}_z)_B (\boldsymbol{\theta}_z)_C}\right) \quad (\text{for } \alpha_1 \text{ and } \alpha_3)$$

The results are given by the Bartlett identity (see (1-11)).

$$(2-12) \quad E\left(\frac{\partial^4 \bar{l}}{\partial(\boldsymbol{\theta}_z)_A \partial(\boldsymbol{\theta}_z)_B (\boldsymbol{\theta}_z)_C (\boldsymbol{\theta}_z)_D}\right) \quad (\text{for } \alpha_{\Delta 2} \text{ and } \alpha_4)$$

The results are given by the Bartlett identity (see (1-12)).

### (3) $\boldsymbol{\mu}$ , $\boldsymbol{\Sigma}$ , $\boldsymbol{\beta}$ , and $\boldsymbol{\xi}$

Some of the expectations of the log likelihood derivatives with respect to

$$\boldsymbol{\theta}_x \text{ and } \boldsymbol{\theta}_z, \text{ e.g., } N^2 E\left(\frac{\partial \bar{l}}{\partial(\boldsymbol{\theta}_x)_A} \frac{\partial \bar{l}}{\partial(\boldsymbol{\theta}_x)_B} \frac{\partial \bar{l}}{\partial(\boldsymbol{\theta}_z)_C} \frac{\partial \bar{l}}{\partial(\boldsymbol{\theta}_z)_D}\right) \text{ are nonzero}$$

though the corresponding fourth cumulants are zero. The results are given as in (1-3), (1-4), (2-3), and (2-4).

## 2. The nonzero partial derivatives of $\boldsymbol{\eta}$ with respect to $\boldsymbol{\theta}$ .

We define  $S = S(\boldsymbol{\xi}, \boldsymbol{\Sigma}) = (1 + \boldsymbol{\xi}' \boldsymbol{\Sigma} \boldsymbol{\xi})^{1/2} (= R^{-1})$ , then

$$\boldsymbol{\tau} = (\boldsymbol{\beta} + \mathbf{1}_{K-1} \boldsymbol{\xi}' \boldsymbol{\mu}) S^{-1} \quad \text{and} \quad \boldsymbol{\rho} = -(\text{Diag } \boldsymbol{\Sigma})^{-1/2} \boldsymbol{\Sigma} \boldsymbol{\xi} S^{-1}.$$

### 2.1 First derivatives

$$\frac{\partial \boldsymbol{\tau}}{\partial \mu_k} = \mathbf{1}_{K-1} \xi_k S^{-1}, \quad \frac{\partial \boldsymbol{\tau}}{\partial \sigma_{ab}} = -\boldsymbol{\tau} S^{-1} \frac{\partial S}{\partial \sigma_{ab}},$$

$$\frac{\partial \boldsymbol{\tau}}{\partial \beta_k} = \mathbf{e}_k S^{-1}, \quad \frac{\partial \boldsymbol{\tau}}{\partial \xi_k} = \mathbf{1}_{K-1} \mu_k S^{-1} - \boldsymbol{\tau} S^{-1} \frac{\partial S}{\partial \xi_k},$$

where  $\mathbf{e}_k$  is the vector of an appropriate dimension whose  $k$ -th element is 1 and the remaining ones are 0.

$$\frac{\partial \boldsymbol{\rho}}{\partial \sigma_{ab}} = \frac{\delta_{ab}}{2} \sigma_{aa}^{-3/2} \mathbf{E}_{aa} \boldsymbol{\Sigma} \boldsymbol{\xi} S^{-1} - \frac{2 - \delta_{ab}}{2} (\text{Diag } \boldsymbol{\Sigma})^{-1/2} (\mathbf{E}_{ab} + \mathbf{E}_{ba}) \boldsymbol{\xi} S^{-1} - \boldsymbol{\rho} S^{-1} \frac{\partial S}{\partial \sigma_{ab}},$$

$$\frac{\partial \boldsymbol{\rho}}{\partial \xi_k} = -(\text{Diag } \boldsymbol{\Sigma})^{-1/2} (\boldsymbol{\Sigma})_{\cdot k} S^{-1} - \boldsymbol{\rho} S^{-1} \frac{\partial S}{\partial \xi_k},$$

where  $\frac{\partial S}{\partial \sigma_{ab}} = \frac{2 - \delta_{ab}}{2} \xi_a \xi_b S^{-1}$ ,  $\frac{\partial S}{\partial \xi_k} = (\boldsymbol{\Sigma} \boldsymbol{\xi})_k S^{-1}$ , and  $\mathbf{E}_{ab}$  is the matrix of an appropriate size whose  $(a, b)$ th element is 1 and the remaining ones are 0.

## 2.2 Second derivatives

$$\frac{\partial^2 \boldsymbol{\tau}}{\partial \sigma_{ab} \partial \mu_k} = -\mathbf{1}_{K-1} \xi_k S^{-2} \frac{\partial S}{\partial \sigma_{ab}},$$

$$\frac{\partial^2 \boldsymbol{\tau}}{\partial \sigma_{ab} \partial \sigma_{cd}} = - \sum_{(ab, cd)}^2 \frac{\partial \boldsymbol{\tau}}{\partial \sigma_{cd}} S^{-1} \frac{\partial S}{\partial \sigma_{ab}} - \boldsymbol{\tau} S^{-1} \frac{\partial^2 S}{\partial \sigma_{ab} \partial \sigma_{cd}},$$

$$\frac{\partial^2 \boldsymbol{\tau}}{\partial \beta_k \partial \sigma_{ab}} = -\mathbf{e}_k S^{-2} \frac{\partial S}{\partial \sigma_{ab}}, \quad \frac{\partial^2 \boldsymbol{\tau}}{\partial \xi_k \partial \mu_l} = \mathbf{1}_{K-1} \left( \delta_{kl} S^{-1} - \xi_l S^{-2} \frac{\partial S}{\partial \xi_k} \right),$$

$$\frac{\partial^2 \boldsymbol{\tau}}{\partial \xi_k \partial \sigma_{ab}} = -\frac{\partial \boldsymbol{\tau}}{\partial \xi_k} S^{-1} \frac{\partial S}{\partial \sigma_{ab}} - \frac{\partial \boldsymbol{\tau}}{\partial \sigma_{ab}} S^{-1} \frac{\partial S}{\partial \xi_k} - \boldsymbol{\tau} S^{-1} \frac{\partial^2 S}{\partial \xi_k \partial \sigma_{ab}},$$

$$\frac{\partial^2 \boldsymbol{\tau}}{\partial \xi_k \partial \beta_l} = -\frac{\partial \boldsymbol{\tau}}{\partial \beta_l} S^{-1} \frac{\partial S}{\partial \xi_k}, \quad \frac{\partial^2 \boldsymbol{\tau}}{\partial \xi_k \partial \xi_l} = -\sum_{(kl)}^2 \frac{\partial \boldsymbol{\tau}}{\partial \xi_l} S^{-1} \frac{\partial S}{\partial \xi_k} - \boldsymbol{\tau} S^{-1} \frac{\partial^2 S}{\partial \xi_k \partial \xi_l},$$

$$\frac{\partial^2 \boldsymbol{\rho}}{\partial \sigma_{ab} \partial \sigma_{cd}} = -\frac{3}{4} \delta_{abcd} \sigma_{aa}^{-5/2} \mathbf{E}_{aa} \boldsymbol{\Sigma} \boldsymbol{\xi} S^{-1} + \sum_{(ab, cd)}^2 \frac{\delta_{ab}}{4} \sigma_{aa}^{-3/2} (2 - \delta_{cd})$$

$$\times (\delta_{ac} \mathbf{E}_{ad} + \delta_{ad} \mathbf{E}_{ac}) \boldsymbol{\xi} S^{-1} - \sum_{(ab, cd)}^2 \frac{\partial \boldsymbol{\rho}}{\partial \sigma_{cd}} S^{-1} \frac{\partial S}{\partial \sigma_{ab}} - \boldsymbol{\rho} S^{-1} \frac{\partial^2 S}{\partial \sigma_{ab} \partial \sigma_{cd}},$$

$$\begin{aligned} \frac{\partial^2 \boldsymbol{\rho}}{\partial \xi_k \partial \sigma_{ab}} &= \frac{\delta_{ab}}{2} \sigma_{aa}^{-3/2} \mathbf{E}_{aa}(\boldsymbol{\Sigma})_{\cdot k} S^{-1} - \frac{2 - \delta_{ab}}{2} (\text{Diag } \boldsymbol{\Sigma})^{-1/2} (\mathbf{e}_a \delta_{bk} + \mathbf{e}_b \delta_{ak}) S^{-1} \\ &\quad - \frac{\partial \boldsymbol{\rho}}{\partial \xi_k} S^{-1} \frac{\partial S}{\partial \sigma_{ab}} - \frac{\partial \boldsymbol{\rho}}{\partial \sigma_{ab}} S^{-1} \frac{\partial S}{\partial \xi_k} - \boldsymbol{\rho} S^{-1} \frac{\partial^2 S}{\partial \xi_k \partial \sigma_{ab}}, \\ \frac{\partial^2 \boldsymbol{\rho}}{\partial \xi_k \partial \xi_l} &= - \sum_{(kl)}^2 \frac{\partial \boldsymbol{\rho}}{\partial \xi_l} S^{-1} \frac{\partial S}{\partial \xi_k} - \boldsymbol{\rho} S^{-1} \frac{\partial^2 S}{\partial \xi_k \partial \xi_l}, \end{aligned}$$

where

$$\begin{aligned} \frac{\partial^2 S}{\partial \sigma_{ab} \partial \sigma_{cd}} &= -\frac{1}{4} (2 - \delta_{ab})(2 - \delta_{cd}) \xi_a \xi_b \xi_c \xi_d S^{-3}, \\ \frac{\partial^2 S}{\partial \xi_k \partial \sigma_{ab}} &= \frac{2 - \delta_{ab}}{2} \left\{ (\delta_{ka} \xi_b + \delta_{kb} \xi_a) S^{-1} - \xi_a \xi_b S^{-2} \frac{\partial S}{\partial \xi_k} \right\}, \\ \frac{\partial^2 S}{\partial \xi_k \partial \xi_l} &= \sigma_{kl} S^{-1} - \frac{\partial S}{\partial \xi_k} \frac{\partial S}{\partial \xi_l} S^{-1}. \end{aligned}$$

### 2.3 Third derivatives

$$\begin{aligned} \frac{\partial^3 \boldsymbol{\tau}}{\partial \sigma_{ab} \partial \sigma_{cd} \partial \mu_k} &= - \sum_{(ab,cd)}^2 \frac{\partial^2 \boldsymbol{\tau}}{\partial \sigma_{cd} \partial \mu_k} S^{-1} \frac{\partial S}{\partial \sigma_{ab}} - \frac{\partial \boldsymbol{\tau}}{\partial \mu_k} S^{-1} \frac{\partial^2 S}{\partial \sigma_{ab} \partial \sigma_{cd}}, \\ \frac{\partial^3 \boldsymbol{\tau}}{\partial \sigma_{ab} \partial \sigma_{cd} \partial \sigma_{ef}} &= - \sum_{(ab,cd,ef)}^3 \left( \frac{\partial^2 \boldsymbol{\tau}}{\partial \sigma_{cd} \partial \sigma_{ef}} S^{-1} \frac{\partial S}{\partial \sigma_{ab}} + \frac{\partial \boldsymbol{\tau}}{\partial \sigma_{cd}} S^{-1} \frac{\partial^2 S}{\partial \sigma_{ab} \partial \sigma_{ef}} \right) \\ &\quad - \boldsymbol{\tau} S^{-1} \frac{\partial^3 S}{\partial \sigma_{ab} \partial \sigma_{cd} \partial \sigma_{ef}}, \\ \frac{\partial^3 \boldsymbol{\tau}}{\partial \beta_k \partial \sigma_{ab} \partial \sigma_{cd}} &= - \sum_{(ab,cd)}^2 \frac{\partial^2 \boldsymbol{\tau}}{\partial \sigma_{cd} \partial \beta_k} S^{-1} \frac{\partial S}{\partial \sigma_{ab}} - \frac{\partial \boldsymbol{\tau}}{\partial \beta_k} S^{-1} \frac{\partial^2 S}{\partial \sigma_{ab} \partial \sigma_{cd}}, \\ \frac{\partial^3 \boldsymbol{\tau}}{\partial \xi_k \partial \sigma_{ab} \partial \mu_l} &= - \frac{\partial^2 \boldsymbol{\tau}}{\partial \xi_k \partial \mu_l} S^{-1} \frac{\partial S}{\partial \sigma_{ab}} - \frac{\partial^2 \boldsymbol{\tau}}{\partial \sigma_{ab} \partial \mu_l} S^{-1} \frac{\partial S}{\partial \xi_k} - \frac{\partial \boldsymbol{\tau}}{\partial \mu_l} S^{-1} \frac{\partial^2 S}{\partial \xi_k \partial \sigma_{ab}}, \end{aligned}$$

$$\begin{aligned}
\frac{\partial^3 \tau}{\partial \xi_k \partial \sigma_{ab} \partial \sigma_{cd}} &= \sum_{(ab,cd)}^2 \left( -\frac{\partial^2 \tau}{\partial \sigma_{cd} \partial \xi_k} S^{-1} \frac{\partial S}{\partial \sigma_{ab}} + \frac{\partial \tau}{\partial \sigma_{cd}} S^{-2} \frac{\partial S}{\partial \xi_k} \frac{\partial S}{\partial \sigma_{ab}} \right. \\
&\quad \left. - \frac{\partial \tau}{\partial \sigma_{cd}} S^{-1} \frac{\partial^2 S}{\partial \sigma_{ab} \partial \xi_k} \right) \\
&\quad - \frac{\partial \tau}{\partial \xi_k} S^{-1} \frac{\partial^2 S}{\partial \sigma_{ab} \partial \sigma_{cd}} + \tau S^{-2} \frac{\partial S}{\partial \xi_k} \frac{\partial^2 S}{\partial \sigma_{ab} \partial \sigma_{cd}} - \tau S^{-1} \frac{\partial^3 S}{\partial \xi_k \partial \sigma_{ab} \partial \sigma_{cd}}, \\
\frac{\partial^3 \tau}{\partial \xi_k \partial \beta_l \partial \sigma_{ab}} &= -\frac{\partial^2 \tau}{\partial \xi_k \partial \beta_l} S^{-1} \frac{\partial S}{\partial \sigma_{ab}} - \frac{\partial^2 \tau}{\partial \beta_l \partial \sigma_{ab}} S^{-1} \frac{\partial S}{\partial \xi_k} - \frac{\partial \tau}{\partial \beta_l} S^{-1} \frac{\partial^2 S}{\partial \xi_k \partial \sigma_{ab}}, \\
\frac{\partial^3 \tau}{\partial \xi_k \partial \xi_l \partial \mu_m} &= -\sum_{(kl)}^2 \frac{\partial^2 \tau}{\partial \xi_l \partial \mu_m} S^{-1} \frac{\partial S}{\partial \xi_k} - \frac{\partial \tau}{\partial \mu_m} S^{-1} \frac{\partial^2 S}{\partial \xi_k \partial \xi_l}, \\
\frac{\partial^3 \tau}{\partial \xi_k \partial \xi_l \partial \sigma_{ab}} &= \sum_{(kl)}^2 \left( -\frac{\partial^2 \tau}{\partial \xi_l \partial \sigma_{ab}} S^{-1} \frac{\partial S}{\partial \xi_k} + \frac{\partial \tau}{\partial \xi_l} S^{-2} \frac{\partial S}{\partial \sigma_{ab}} \frac{\partial S}{\partial \xi_k} \right. \\
&\quad \left. - \frac{\partial \tau}{\partial \xi_l} S^{-1} \frac{\partial^2 S}{\partial \xi_k \partial \sigma_{ab}} \right) \\
&\quad - \frac{\partial \tau}{\partial \sigma_{ab}} S^{-1} \frac{\partial^2 S}{\partial \xi_k \partial \xi_l} + \tau S^{-2} \frac{\partial S}{\partial \sigma_{ab}} \frac{\partial^2 S}{\partial \xi_k \partial \xi_l} - \tau S^{-1} \frac{\partial^3 S}{\partial \xi_k \partial \xi_l \partial \sigma_{ab}}, \\
\frac{\partial^3 \tau}{\partial \xi_k \partial \xi_l \partial \beta_m} &= -\sum_{(kl)}^2 \frac{\partial^2 \tau}{\partial \xi_l \partial \beta_m} S^{-1} \frac{\partial S}{\partial \xi_k} - \frac{\partial \tau}{\partial \beta_m} S^{-1} \frac{\partial^2 S}{\partial \xi_k \partial \xi_l}, \\
\frac{\partial^3 \tau}{\partial \xi_k \partial \xi_l \partial \xi_m} &= -\sum_{(klm)}^3 \left( \frac{\partial^2 \tau}{\partial \xi_l \partial \xi_m} S^{-1} \frac{\partial S}{\partial \xi_k} + \frac{\partial \tau}{\partial \xi_l} S^{-1} \frac{\partial^2 S}{\partial \xi_k \partial \xi_m} \right) \\
&\quad - \tau S^{-1} \frac{\partial^3 S}{\partial \xi_k \partial \xi_l \partial \xi_m}, \\
\frac{\partial^3 \rho}{\partial \sigma_{ab} \partial \sigma_{cd} \partial \sigma_{ef}} &= \frac{15}{8} \delta_{abcdef} \sigma_{aa}^{-7/2} \mathbf{E}_{aa} \Sigma \xi S^{-1} + \sum_{(ab,cd,ef)}^3 \left\{ -\frac{3}{8} \delta_{abcd} \sigma_{aa}^{-5/2} (2 - \delta_{ef}) \right. \\
&\quad \left. \times (\delta_{ae} \mathbf{E}_{af} + \delta_{af} \mathbf{E}_{ae}) \xi S^{-1} - \frac{\partial^2 \rho}{\partial \sigma_{cd} \partial \sigma_{ef}} S^{-1} \frac{\partial S}{\partial \sigma_{ab}} - \frac{\partial \rho}{\partial \sigma_{ef}} S^{-1} \frac{\partial^2 S}{\partial \sigma_{ab} \partial \sigma_{cd}} \right\}
\end{aligned}$$

$$\begin{aligned}
& -\boldsymbol{\rho} S^{-1} \frac{\partial^3 S}{\partial \sigma_{ab} \partial \sigma_{cd} \partial \sigma_{ef}}, \\
& \frac{\partial^3 \boldsymbol{\rho}}{\partial \xi_k \partial \sigma_{ab} \partial \sigma_{cd}} = -\frac{3}{4} \delta_{abcd} \sigma_{aa}^{-5/2} \mathbf{E}_{aa} \left\{ (\boldsymbol{\Sigma})_{\cdot k} S^{-1} - \boldsymbol{\Sigma} \xi S^{-2} \frac{\partial S}{\partial \xi_k} \right\} + \sum_{(ab,cd)}^2 \frac{\sigma_{ab}}{4} \sigma_{aa}^{-3/2} \\
& \times (2 - \delta_{cd}) \left\{ (\delta_{ac} \delta_{dk} + \delta_{ad} \delta_{ck}) \mathbf{e}_a S^{-1} - (\delta_{ac} \mathbf{E}_{ad} + \delta_{ad} \mathbf{E}_{ac}) \xi S^{-2} \frac{\partial S}{\partial \xi_k} \right\} \\
& + \sum_{(ab,cd)}^2 \left( -\frac{\partial^2 \boldsymbol{\rho}}{\partial \xi_k \partial \sigma_{cd}} S^{-1} \frac{\partial S}{\partial \sigma_{ab}} + \frac{\partial \boldsymbol{\rho}}{\partial \sigma_{cd}} S^{-2} \frac{\partial S}{\partial \xi_k} \frac{\partial S}{\partial \sigma_{ab}} - \frac{\partial \boldsymbol{\rho}}{\partial \sigma_{cd}} S^{-1} \frac{\partial^2 S}{\partial \xi_k \partial \sigma_{ab}} \right) \\
& - \frac{\partial \boldsymbol{\rho}}{\partial \xi_k} S^{-1} \frac{\partial^2 S}{\partial \sigma_{ab} \partial \sigma_{cd}} + \boldsymbol{\rho} S^{-2} \frac{\partial S}{\partial \xi_k} \frac{\partial^2 S}{\partial \sigma_{ab} \partial \sigma_{cd}} - \boldsymbol{\rho} S^{-1} \frac{\partial^3 S}{\partial \xi_k \partial \sigma_{ab} \partial \sigma_{cd}}, \\
& \frac{\partial^3 \boldsymbol{\rho}}{\partial \xi_k \partial \xi_l \partial \sigma_{ab}} = \sum_{(kl)}^2 \left( -\frac{\partial^2 \boldsymbol{\rho}}{\partial \xi_l \partial \sigma_{ab}} S^{-1} \frac{\partial S}{\partial \xi_k} + \frac{\partial \boldsymbol{\rho}}{\partial \xi_l} S^{-2} \frac{\partial S}{\partial \sigma_{ab}} \frac{\partial S}{\partial \xi_k} \right. \\
& \quad \left. - \frac{\partial \boldsymbol{\rho}}{\partial \xi_l} S^{-1} \frac{\partial^2 S}{\partial \xi_k \partial \sigma_{ab}} \right) \\
& - \frac{\partial \boldsymbol{\rho}}{\partial \sigma_{ab}} S^{-1} \frac{\partial^2 S}{\partial \xi_k \partial \xi_l} + \boldsymbol{\rho} S^{-2} \frac{\partial S}{\partial \sigma_{ab}} \frac{\partial^2 S}{\partial \xi_k \partial \xi_l} - \boldsymbol{\rho} S^{-1} \frac{\partial^3 S}{\partial \xi_k \partial \xi_l \partial \sigma_{ab}}, \\
& \frac{\partial^3 \boldsymbol{\rho}}{\partial \xi_k \partial \xi_l \partial \xi_m} = -\sum_{(klm)}^3 \left( \frac{\partial^2 \boldsymbol{\rho}}{\partial \xi_l \partial \xi_m} S^{-1} \frac{\partial S}{\partial \xi_k} + \frac{\partial \boldsymbol{\rho}}{\partial \xi_l} S^{-1} \frac{\partial^2 S}{\partial \xi_k \partial \xi_m} \right) - \boldsymbol{\rho} S^{-1} \frac{\partial^3 S}{\partial \xi_k \partial \xi_l \partial \xi_m},
\end{aligned}$$

where

$$\frac{\partial^3 S}{\partial \sigma_{ab} \partial \sigma_{cd} \partial \sigma_{ef}} = \frac{3}{8} (2 - \delta_{ab})(2 - \delta_{cd})(2 - \delta_{ef}) \xi_a \xi_b \xi_c \xi_d \xi_e \xi_f S^{-5},$$

$$\begin{aligned}
\frac{\partial^3 S}{\partial \xi_k \partial \sigma_{ab} \partial \sigma_{cd}} &= (2 - \delta_{ab})(2 - \delta_{cd}) \left[ \sum_{(ab,cd)}^2 \left\{ -\frac{1}{4} (\delta_{ka} \xi_b + \delta_{kb} \xi_a) \xi_c \xi_d S^{-3} \right\} \right. \\
&\quad \left. + \frac{3}{4} \xi_a \xi_b \xi_c \xi_d S^{-4} \frac{\partial S}{\partial \xi_k} \right],
\end{aligned}$$

$$\begin{aligned} \frac{\partial^3 S}{\partial \xi_k \partial \xi_l \partial \sigma_{ab}} &= \frac{2 - \delta_{ab}}{2} (\delta_{ka} \delta_{lb} + \delta_{kb} \delta_{la}) S^{-1} - \sigma_{kl} S^{-2} \frac{\partial S}{\partial \sigma_{ab}} \\ &\quad - \sum_{(kl)}^2 \frac{\partial^2 S}{\partial \xi_k \partial \sigma_{ab}} \frac{\partial S}{\partial \xi_l} S^{-1} + \frac{\partial S}{\partial \xi_k} \frac{\partial S}{\partial \xi_l} S^{-2} \frac{\partial S}{\partial \sigma_{ab}}, \\ \frac{\partial^3 S}{\partial \xi_k \partial \xi_l \partial \xi_m} &= - \sum_{(klm)}^3 \frac{\partial^2 S}{\partial \xi_k \partial \xi_m} \frac{\partial S}{\partial \xi_l} S^{-1}. \end{aligned}$$

### 3. Computation by Gaussian quadrature

#### 3.1 Univariate case

Stroud and Sechrest (1966, Table Five, pp.217-252) gave the following values of  $A_i$  and  $x_i$ :

$$\int_{-\infty}^{\infty} \exp(-x^2) f(x) dx \approx \sum_{i=1}^n A_i f(x_i),$$

where  $n=2(1)64(4)96(8)136$ . Let  $y = \sqrt{2} x$ . Then,

$$\begin{aligned} \int_{-\infty}^{\infty} \phi(y) f(y) dy &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-y^2 / 2) f(y) dy \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} \exp(-x^2) f(\sqrt{2} x) dx \approx \frac{1}{\sqrt{\pi}} \sum_{i=1}^n A_i f(\sqrt{2} x_i). \end{aligned}$$

The above result corresponds to Bock and Lieberman (1970, Equation (5)).

Bock and Lieberman (1970, p.183) used  $n=64$ , where only 40 values of  $x_i$  were employed since  $A_i$ 's for the remaining  $x_i$ 's are  $A_i < 10^{-14}$ . Bock and Aitkin (1981, Table 1) reported the results using  $n=10$  and 2.

#### 3.2 Bivariate case

Suppose that  $\mathbf{U} = (X, Y)' \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ . Let the density of  $\mathbf{U}$  at  $\mathbf{u} = (x, y)'$  be

$$\phi_2(x, y) = \frac{1}{2\pi |\boldsymbol{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{u} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1}(\mathbf{u} - \boldsymbol{\mu})\right\},$$

where  $\Sigma = \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{yx} & \sigma_y^2 \end{pmatrix}$ ,  $\boldsymbol{\mu} = (\mu_x, \mu_y)'$  and  $\phi_2(\mathbf{U} = \mathbf{u}) = \phi_2(x, y)$

$= \phi_2(x, y, \boldsymbol{\mu}, \Sigma)$ . Define  $\phi_1(x, \mu, \sigma_x^2) = \frac{1}{\sqrt{2\pi\sigma_x}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma_x^2}\right\}$  (note

$\phi_1(x, 0, 1) = \phi(x)$ ). Then, using the transformations

$z_x = \frac{x - \mu_x}{\sigma_x}$  and  $z_{y|x} = \frac{y - \mu_y - \{(\sigma_{xy} / \sigma_x^2)\}(x - \mu_x)}{[\sigma_y^2 - \{(\sigma_{xy})^2 / \sigma_x^2\}]^{1/2}} \equiv \frac{y - \mu_{y|x}}{\sigma_{y|x}}$ , it follows

that

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_2(x, y) g(x, y) dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_1(x, \mu_x, \sigma_x^2) \\ & \quad \times \phi_1[y, \mu_y + (\sigma_{xy} / \sigma_x^2)(x - \mu_x), \sigma_y^2 - \{(\sigma_{xy})^2 / \sigma_x^2\}] g(x, y) dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(z_x) \phi(z_{y|x}) g(\mu_x + \sigma_x z_x, \mu_{y|x} + \sigma_{y|x} z_{y|x}) dz_x dz_{y|x} \\ &\simeq \int_{-\infty}^{\infty} \phi(z_x) \frac{1}{\sqrt{\pi}} \sum_{j=1}^n A_j g(\mu_x + \sigma_x z_x, \mu_{y|x} + \sigma_{y|x} \sqrt{2} x_j) dz_x \\ &\simeq \frac{1}{\pi} \sum_{i=1}^n \sum_{j=1}^n A_i A_j g(\mu_x + \sigma_x \sqrt{2} x_i, \mu_{y|x}^* + \sigma_{y|x} \sqrt{2} x_j) dz_x, \end{aligned}$$

where  $\mu_{y|x}^* = \mu_{y|\mu_x + \sigma_x \sqrt{2} x_i} = \mu_y + (\sigma_{xy} / \sigma_x) \sqrt{2} x_i$ .

## Part B

### 1. The partial derivatives of $\tilde{F}_{\text{ULS}}$ with respect to $\hat{\theta}$

$$\frac{\partial \tilde{F}_{\text{ULS}}}{\partial \hat{\theta}_i} = \text{tr} \left\{ (\hat{\Sigma} - \mathbf{S}) \frac{\partial \hat{\Sigma}}{\partial \hat{\theta}_i} \right\}, \quad \frac{\partial^2 \tilde{F}_{\text{ULS}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} = \text{tr} \left\{ (\hat{\Sigma} - \mathbf{S}) \frac{\partial^2 \hat{\Sigma}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} + \frac{\partial \hat{\Sigma}}{\partial \hat{\theta}_i} \frac{\partial \hat{\Sigma}}{\partial \hat{\theta}_j} \right\},$$

$$\frac{\partial^2 \tilde{F}_{\text{ULS}}}{\partial \hat{\theta}_i \partial s_{ab}} = -(2 - \delta_{ab}) \left( \frac{\partial \hat{\Sigma}}{\partial \hat{\theta}_i} \right)_{ab},$$

$$\frac{\partial^3 \tilde{F}_{\text{ULS}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k} = \text{tr} \left\{ (\hat{\Sigma} - \mathbf{S}) \frac{\partial^3 \hat{\Sigma}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k} + \sum^3 \frac{\partial \hat{\Sigma}}{\partial \hat{\theta}_i} \frac{\partial^2 \hat{\Sigma}}{\partial \hat{\theta}_j \partial \hat{\theta}_k} \right\},$$

$$\frac{\partial^3 \tilde{F}_{\text{ULS}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial s_{ab}} = -(2 - \delta_{ab}) \left( \frac{\partial^2 \hat{\Sigma}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \right)_{ab},$$

$$\begin{aligned} \frac{\partial^4 \tilde{F}_{\text{ULS}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k \partial \hat{\theta}_l} &= \text{tr} \left\{ (\hat{\Sigma} - \mathbf{S}) \frac{\partial^4 \hat{\Sigma}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k \partial \hat{\theta}_l} \right. \\ &\quad \left. + \sum^4 \frac{\partial \hat{\Sigma}}{\partial \hat{\theta}_i} \frac{\partial^3 \hat{\Sigma}}{\partial \hat{\theta}_j \partial \hat{\theta}_k \partial \hat{\theta}_l} + \sum^3 \frac{\partial^2 \hat{\Sigma}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \frac{\partial^2 \hat{\Sigma}}{\partial \hat{\theta}_k \partial \hat{\theta}_l} \right\}, \end{aligned}$$

$$\frac{\partial^4 \tilde{F}_{\text{ULS}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k \partial s_{ab}} = -(2 - \delta_{ab}) \left( \frac{\partial^3 \hat{\Sigma}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k} \right)_{ab},$$

$$\begin{aligned} \frac{\partial^5 \tilde{F}_{\text{ULS}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k \partial \hat{\theta}_l \partial \hat{\theta}_m} &= \text{tr} \left\{ (\hat{\Sigma} - \mathbf{S}) \frac{\partial^5 \hat{\Sigma}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k \partial \hat{\theta}_l \partial \hat{\theta}_m} \right. \\ &\quad \left. + \sum^5 \frac{\partial \hat{\Sigma}}{\partial \hat{\theta}_i} \frac{\partial^4 \hat{\Sigma}}{\partial \hat{\theta}_j \partial \hat{\theta}_k \partial \hat{\theta}_l \partial \hat{\theta}_m} + \sum^{10} \frac{\partial^2 \hat{\Sigma}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \frac{\partial^3 \hat{\Sigma}}{\partial \hat{\theta}_k \partial \hat{\theta}_l \partial \hat{\theta}_m} \right\}, \end{aligned}$$

$$\frac{\partial^5 \tilde{F}_{\text{ULS}}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k \partial \hat{\theta}_l \partial s_{ab}} = -(2 - \delta_{ab}) \left( \frac{\partial^4 \hat{\Sigma}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k \partial \hat{\theta}_l} \right)_{ab}$$

( $i, j, k, l, m = 1, \dots, q; p \geq a \geq b \geq 1$ ).

Using the above results with Lemma 1, we have the partial derivatives of  $\hat{F}_{\text{ULS}}$  with respect to  $\mathbf{s}$  as follows:

$$\frac{\partial \hat{F}_{\text{ULS}}}{\partial s_{ab}} = \text{tr} \left\{ (\hat{\Sigma} - \mathbf{S}) \frac{\partial \hat{\Sigma}}{\partial \hat{\theta}_i} \right\} \frac{\partial \hat{\theta}_i}{\partial s_{ab}} - (2 - \delta_{ab}) (\hat{\Sigma} - \mathbf{S})_{ab},$$

$$\begin{aligned} \frac{\partial^2 \hat{F}_{\text{ULS}}}{\partial s_{ab} \partial s_{cd}} &= \text{tr} \left\{ \frac{\partial \hat{\Sigma}}{\partial \hat{\theta}_i} \frac{\partial \hat{\Sigma}}{\partial \hat{\theta}_j} + (\hat{\Sigma} - \mathbf{S}) \frac{\partial^2 \hat{\Sigma}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \right\} \frac{\partial \hat{\theta}_i}{\partial s_{ab}} \frac{\partial \hat{\theta}_j}{\partial s_{cd}} \\ &+ \text{tr} \left\{ (\hat{\Sigma} - \mathbf{S}) \frac{\partial \hat{\Sigma}}{\partial \hat{\theta}_i} \right\} \frac{\partial^2 \hat{\theta}_i}{\partial s_{ab} \partial s_{cd}} - \sum^2 (2 - \delta_{ab}) \frac{\partial \hat{\sigma}_{ab}}{\partial \hat{\theta}_i} \frac{\partial \hat{\theta}_i}{\partial s_{cd}} \\ &+ (2 - \delta_{ab}) \delta_{ac} \delta_{bd}, \end{aligned}$$

$$\begin{aligned} \frac{\partial^3 \hat{F}_{\text{ULS}}}{\partial s_{ab} \partial s_{cd} \partial s_{ef}} &= \text{tr} \left\{ \sum^3 \frac{\partial \hat{\Sigma}}{\partial \hat{\theta}_i} \frac{\partial^2 \hat{\Sigma}}{\partial \hat{\theta}_j \partial \hat{\theta}_k} + (\hat{\Sigma} - \mathbf{S}) \frac{\partial^3 \hat{\Sigma}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k} \right\} \frac{\partial \hat{\theta}_i}{\partial s_{ab}} \frac{\partial \hat{\theta}_j}{\partial s_{cd}} \frac{\partial \hat{\theta}_k}{\partial s_{ef}} \\ &+ \text{tr} \left\{ \frac{\partial \hat{\Sigma}}{\partial \hat{\theta}_i} \frac{\partial \hat{\Sigma}}{\partial \hat{\theta}_j} + (\hat{\Sigma} - \mathbf{S}) \frac{\partial^2 \hat{\Sigma}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \right\} \sum^3 \frac{\partial \hat{\theta}_i}{\partial s_{ab}} \frac{\partial^2 \hat{\theta}_j}{\partial s_{cd} \partial s_{ef}} \\ &+ \text{tr} \left\{ (\hat{\Sigma} - \mathbf{S}) \frac{\partial \hat{\Sigma}}{\partial \hat{\theta}_i} \right\} \frac{\partial^3 \hat{\theta}_i}{\partial s_{ab} \partial s_{cd} \partial s_{ef}} \\ &- \sum^3 (2 - \delta_{ab}) \left( \frac{\partial^2 \hat{\sigma}_{ab}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \frac{\partial \hat{\theta}_i}{\partial s_{cd}} \frac{\partial \hat{\theta}_j}{\partial s_{ef}} + \frac{\partial \hat{\sigma}_{ab}}{\partial \hat{\theta}_i} \frac{\partial^2 \hat{\theta}_i}{\partial s_{cd} \partial s_{ef}} \right), \end{aligned}$$

$$\frac{\partial^4 \hat{F}_{\text{ULS}}}{\partial s_{ab} \partial s_{cd} \partial s_{ef} \partial s_{gh}} = \text{tr} \left\{ \sum^3 \frac{\partial^2 \hat{\Sigma}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \frac{\partial^2 \hat{\Sigma}}{\partial \hat{\theta}_k \partial \hat{\theta}_l} + \sum^4 \frac{\partial \hat{\Sigma}}{\partial \hat{\theta}_i} \frac{\partial^3 \hat{\Sigma}}{\partial \hat{\theta}_j \partial \hat{\theta}_k \partial \hat{\theta}_l} \right\}$$

$$\begin{aligned}
& +(\hat{\Sigma} - \mathbf{S}) \frac{\partial^4 \hat{\Sigma}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k \partial \hat{\theta}_l} \left\{ \frac{\partial \hat{\theta}_i}{\partial s_{ab}} \frac{\partial \hat{\theta}_j}{\partial s_{cd}} \frac{\partial \hat{\theta}_k}{\partial s_{ef}} \frac{\partial \hat{\theta}_l}{\partial s_{gh}} \right. \\
& + \text{tr} \left\{ \sum^3 \frac{\partial \hat{\Sigma}}{\partial \hat{\theta}_i} \frac{\partial^2 \hat{\Sigma}}{\partial \hat{\theta}_j \partial \hat{\theta}_k} + (\hat{\Sigma} - \mathbf{S}) \frac{\partial^3 \hat{\Sigma}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k} \right\} \sum^6 \frac{\partial \hat{\theta}_i}{\partial s_{ab}} \frac{\partial \hat{\theta}_j}{\partial s_{cd}} \frac{\partial^2 \hat{\theta}_k}{\partial s_{ef} \partial s_{gh}} \\
& + \text{tr} \left\{ \sum^3 \frac{\partial \hat{\Sigma}}{\partial \hat{\theta}_i} \frac{\partial \hat{\Sigma}}{\partial \hat{\theta}_j} + (\hat{\Sigma} - \mathbf{S}) \frac{\partial^2 \hat{\Sigma}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \right\} \left( \sum^3 \frac{\partial^2 \hat{\theta}_i}{\partial s_{ab} \partial s_{cd}} \frac{\partial^2 \hat{\theta}_j}{\partial s_{ef} \partial s_{gh}} \right. \\
& \left. + \sum^4 \frac{\partial \hat{\theta}_i}{\partial s_{ab}} \frac{\partial^3 \hat{\theta}_j}{\partial s_{cd} \partial s_{ef} \partial s_{gh}} \right) + \text{tr} \left\{ (\hat{\Sigma} - \mathbf{S}) \frac{\partial \hat{\Sigma}}{\partial \hat{\theta}_i} \right\} \frac{\partial^4 \hat{\theta}_i}{\partial s_{ab} \partial s_{cd} \partial s_{ef} \partial s_{gh}} \\
& - \sum^4 (2 - \delta_{ab}) \left( \frac{\partial^3 \hat{\sigma}_{ab}}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k} \frac{\partial \hat{\theta}_i}{\partial s_{cd}} \frac{\partial \hat{\theta}_j}{\partial s_{ef}} \frac{\partial \hat{\theta}_k}{\partial s_{gh}} + \frac{\partial^2 \hat{\sigma}_{ab}}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \sum^3 \frac{\partial \hat{\theta}_i}{\partial s_{cd}} \frac{\partial^2 \hat{\theta}_j}{\partial s_{ef} \partial s_{gh}} \right. \\
& \left. + \frac{\partial \hat{\sigma}_{ab}}{\partial \hat{\theta}_i} \frac{\partial^3 \hat{\theta}_i}{\partial s_{cd} \partial s_{ef} \partial s_{gh}} \right)
\end{aligned}$$

$$(p \geq a \geq b \geq 1, p \geq c \geq d \geq 1, p \geq e \geq f \geq 1, p \geq g \geq h \geq 1),$$

where Einstein's summation convention is to be used only for subscripts  $i, j, k$  and  $l$ .

## 2. The chain rules

$$\frac{\partial \hat{g}}{\partial s_{ab}} = \frac{\partial \hat{g}}{\partial \hat{F}} \frac{\partial \hat{F}}{\partial s_{ab}},$$

$$\frac{\partial^2 \hat{g}}{\partial s_{ab} \partial s_{cd}} = \frac{\partial^2 \hat{g}}{\partial \hat{F}^2} \frac{\partial \hat{F}}{\partial s_{ab}} \frac{\partial \hat{F}}{\partial s_{cd}} + \frac{\partial \hat{g}}{\partial \hat{F}} \frac{\partial^2 \hat{F}}{\partial s_{ab} \partial s_{cd}},$$

$$\begin{aligned}
& \frac{\partial^3 \hat{g}}{\partial s_{ab} \partial s_{cd} \partial s_{ef}} = \frac{\partial^3 \hat{g}}{\partial \hat{F}^3} \frac{\partial \hat{F}}{\partial s_{ab}} \frac{\partial \hat{F}}{\partial s_{cd}} \frac{\partial \hat{F}}{\partial s_{ef}} \\
& + \sum^3 \frac{\partial^2 \hat{g}}{\partial \hat{F}^2} \frac{\partial \hat{F}}{\partial s_{ab}} \frac{\partial^2 \hat{F}}{\partial s_{cd} \partial s_{ef}} + \frac{\partial \hat{g}}{\partial \hat{F}} \frac{\partial^3 \hat{F}}{\partial s_{ab} \partial s_{cd} \partial s_{ef}}, \\
& \frac{\partial^4 \hat{g}}{\partial s_{ab} \partial s_{cd} \partial s_{ef} \partial s_{gh}} = \frac{\partial^4 \hat{g}}{\partial \hat{F}^4} \frac{\partial \hat{F}}{\partial s_{ab}} \frac{\partial \hat{F}}{\partial s_{cd}} \frac{\partial \hat{F}}{\partial s_{ef}} \frac{\partial \hat{F}}{\partial s_{gh}} \\
& + \sum^6 \frac{\partial^3 \hat{g}}{\partial \hat{F}^3} \frac{\partial \hat{F}}{\partial s_{ab}} \frac{\partial \hat{F}}{\partial s_{cd}} \frac{\partial^2 \hat{F}}{\partial s_{ef} \partial s_{gh}} + \sum^3 \frac{\partial^2 \hat{g}}{\partial \hat{F}^2} \frac{\partial^2 \hat{F}}{\partial s_{ab} \partial s_{cd}} \frac{\partial^2 \hat{F}}{\partial s_{ef} \partial s_{gh}} \\
& + \sum^4 \frac{\partial^2 \hat{g}}{\partial \hat{F}^2} \frac{\partial \hat{F}}{\partial s_{ab}} \frac{\partial^3 \hat{F}}{\partial s_{cd} \partial s_{ef} \partial s_{gh}} + \frac{\partial \hat{g}}{\partial \hat{F}} \frac{\partial^4 \hat{F}}{\partial s_{ab} \partial s_{cd} \partial s_{ef} \partial s_{gh}} \\
& (p \geq a \geq b \geq 1, p \geq c \geq d \geq 1, p \geq e \geq f \geq 1, p \geq g \geq h \geq 1).
\end{aligned}$$

### References

- Bock, R. D., & Lieberman, M. (1970). Fitting a response model for  $n$  dichotomously scored items. *Psychometrika*, 35, 179-197.
- Bock, R. D., & Aitkin, M. (1981). Marginal maximum likelihood estimation of item parameters: Application of an EM algorithm. *Psychometrika*, 46, 443-459.
- Ogasawara, H. (2011). Asymptotic expansions of the distributions of the polyserial correlation coefficients. *Behaviormetrika*, 38 (2), 153-168.
- Ogasawara, H. (2010). Asymptotic expansions of the null distributions of discrepancy functions for general covariance structures under nonnormality. *American Journal of Mathematical and Management Sciences*, 30 (3 &4), 385-422.
- Stroud, A. H., & Secrest, D. (1966). *Gaussian quadrature formulas*. Englewood Cliffs, NJ: Prentice-Hall.